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TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE FLOWS PAST ARBITRARY BODIES BY THE VARIATIONAL METHOD

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SUMMARY

Instead of solving the nonlinear differential equation which governs the compressible flow, an approximate method of solution by means of the variational method is used in this report. The general problem of steady irrotational flow past an arbitrary body is formulated. Two examples were carried out, namely, the flow past a circular cylinder and the flow past a thin curved surface. The variational method yields results of velocity and pressure distributions which compare excellently with those found by existing methods. These results indicate that the variational method will yield good approximate solution for flow past both thick and thin bodies, at both high and low Mach numbers.

INTRODUCTION

The advance in aerodynamics has been greatly accelerated by the interpretation of good experimental results in the light of theoretical predictions. When the flow velocity is small, the change in density of the fluid in the flow is so small that the fluid may be considered to be incompressible. Many useful theoretical results have been obtained from a consideration of the two-dimensional irrotational flow of an incompressible perfect fluid. When the flow velocity becomes large, the change in density of the fluid in the flow can no longer be considered small and the flow must be considered to be compressible. While the two-dimensional incompressible flow can be described by the well-known Laplace equation, the corresponding compressible flow can only be described by a complicated nonlinear differential equation. Except for a few simple cases, the exact solution of this differential equation is still unknown.

In the case of two-dimensional irrotational subsonic flow, there are at least three different methods of small perturbation or linearization. The small-perturbation methods have been carried out by developing the velocity potential in terms of the Mach number or of a thickness parameter. Obviously, the development in terms of the Mach number limits

the applicability of the method to flows at small velocities and the development in terms of the thickness parameter limits the applicability of the method only to flows about very thin bodies. Methods for calculating the second- and the third-order approximations in these methods have been carried out in some cases, but the increased mathematical difficulty with each approximation prevents their application to the higher orders of approximation. Another method of linearization is the so-called hodograph method wherein the nonlinear differential equation is transformed into an exact linear differential equation, thus removing the difficulty of the nonlinearity of the governing equation. However, in doing so, the fulfillment of the boundary conditions is made much more difficult. The application of this method to actual problems requires some simplifying assumptions which can be shown as being only approximately correct.

The variational method, as carried out in this report, on the other hand, can be used to study the compressible flow without linearization and the computation can be made to obtain a solution of high accuracy without any mathematical difficulty. The variational method has also the advantage of being a direct method by means of which one may be able to calculate the flow passing a given airfoil at different Mach numbers. In the variational method, instead of solving the nonlinear differential equations of motion, a variational integral is first formulated and approximate solutions are then calculated by the Rayleigh-Ritz procedure.

Two numerical examples are carried out in this report to ascertain the applicability of this method. They are for the two-dimensional irrotational flows past a circular cylinder and a thin bump as proposed by Kaplan. (See reference 1.) These examples are chosen because in the case of circular cylinders similar results by the Rayleigh-Janzen method are available, and in the case of the Kaplan bump results by means of the perturbation method in terms of thickness parameter are available. A comparison of the results by the present method with those computed by the other methods shows excellent agreement. Since the Rayleigh-Janzen method is known to give good results for thick bodies at a relatively low Mach number and the perturbation method in terms of thickness parameter yields accurate results for thin bodies up to high Mach numbers, the results of this report indicate that the variational method gives good approximate solutions for both thick and thin bodies and at both high and low Mach numbers.

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SYMBOLS

a	velocity of sound
C_p	pressure coefficient
$C_{p,i}$	pressure coefficient for incompressible flow
I	variational integral
M	Mach number (q/a)
n	inward normal direction
p	pressure
$\overline{q}$	velocity vector
q	magnitude of velocity vector
q_{max}	maximum possible velocity of flow
r	position radius in flow field
u	velocity component in x-direction
U	velocity of undisturbed stream
v	velocity component in y-direction
x,y	rectangular coordinates of a point in the fluid
X,Y	coordinates of curved surface divided by semichord of shape

$z = x + iy = re^{i\theta}$

t	thickness parameter
d	constant defining thickness of Kaplan's bump
γ	ratio of specific heats at constant pressure and constant volume
θ	angular position
ϕ	velocity potential

ρ mass density of fluid

ξ,η rectangular coordinates in plane of curved surface

$\zeta = \xi + i\eta$

Γ strength of circulation

Subscripts:

x,y differentiation in corresponding direction

o conditions of undisturbed stream

THEORY

For steady two-dimensional irrotational flows, there are four hydrodynamical variables: The pressure p, the density ρ, and the two components of velocity u and v. All these are to be determined as functions of the coordinates x and y by the following four equations:

(1) The equation of continuity (one equation)

$$\operatorname{div}(\rho\overline{q}) = 0$$

or

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{1}$$

where $\overline{q}$ is the velocity vector.

(2) The equations of motion (two equations)

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\rho^{-1}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\rho^{-1}\frac{\partial p}{\partial y} \end{aligned}\right\} \tag{2}$$

(3) The equation of state (one equation)

$$p = A + B\rho^{\gamma} \tag{3}$$

where A, B, and γ are constants and the significance of A and B will be discussed later.

Most aerodynamic problems are concerned with either flow from rest or flow which is parallel and uniform at infinity. Both of these types are irrotational to begin with, and will remain so provided that friction and shock waves are neglected. Thus, irrotational flow is of great interest in aeronautical applications.

The condition of irrotationality is

$$\operatorname{rot} \overline{q} = 0$$

or

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

This condition can be satisfied by introducing a velocity potential ϕ such that

$$\left.\begin{aligned} u &= \frac{\partial \phi}{\partial x} \\ v &= \frac{\partial \phi}{\partial y} \end{aligned}\right\} \tag{4}$$

that is, q is a gradient of ϕ.

For the velocity field given by equations (4) to represent a physically possible field of flow, it is necessary that the continuity equation (1) be also satisfied. By means of equations (1), (2), and (3), p and ρ are eliminated, and one obtains the following fundamental differential equation:

$$\left(a^2 - \phi_x^{\ 2}\right)\phi_{xx} + \left(a^2 - \phi_y^{\ 2}\right)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} = 0 \tag{5}$$

in which

$$a^2 = a_o^{\ 2} + \frac{\gamma - 1}{2}\left(q_o^{\ 2} - \phi_x^{\ 2} - \phi_y^{\ 2}\right)$$

where a is the velocity of sound in the fluid and the subscript o refers to quantities at infinity. Equation (5) is a nonlinear differential equation, and its exact solution is found to be extremely difficult. Instead of solving the differential equation exactly, an approximate solution may be obtained by means of the variational method. In this method the first step is to formulate a variational principle.

A variational principle for the compressible fluid was first formulated by Hargreaves. (See reference 2.) A somewhat more detailed discussion of this variational principle was made later by Bateman. (See reference 3.) A simplified proof of the principle was given by Wang. (See reference 4.) It seems, however, that the following formulation is more complete from the mathematical point of view and is therefore included here.

Consider the following variational integral

$$I = \int_D \left\{ f(\rho) - \rho \left[\left(u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} \right) - \frac{1}{2}\left(u^2 + v^2\right) \right] \right\} dx\, dy \tag{6}$$

where the integration is taken over the given domain D. Varying ρ, u, v, and ϕ independently and setting the first variation equal to zero, one obtains

$$\delta I = \int_D \left\{ \left[f'(\rho) - \left(u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} \right) + \frac{1}{2}(u^2 + v^2) \right] \delta\rho - \rho\left(\frac{\partial \phi}{\partial x} - u\right)\delta u - \rho\left(\frac{\partial \phi}{\partial y} - v\right)\delta v + \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}\right)\delta\phi \right\} dx\, dy + \int_S \rho\,\delta\phi\, q_n\, dS = 0$$

where S is the boundary of the domain D and q_n is the velocity component which is normal to the boundary S.

The Euler's equations of the integral I, according to the calculus of variations, are then

$$f'(\rho) - \left(u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} \right) + \frac{1}{2}(u^2 + v^2) = 0 \tag{7}$$

$$\frac{\partial \phi}{\partial x} - u = 0 \tag{8}$$

$$\frac{\partial \phi}{\partial y} - v = 0 \tag{9}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{10}$$

and the natural boundary condition must be such that

$$\int_S \rho \delta \phi q_n \, dS = 0 \tag{11}$$

Equation (10) is the continuity equation (1) and equations (8) and (9) are the same as equations (4), which implies that the condition of irrotationality is satisfied. If the pressure p is given by the equation

$$p = -\rho f'(\rho) + f(\rho) \tag{12}$$

equation (7), when combined with equations (8), (9), and (10), indicates that equations (2) are satisfied.

Thus the variational principle (6) leads to the fundamental equations of a two-dimensional compressible fluid if p is of the form given by equation (12) and the condition (11) is satisfied. Equation (12) is a differential equation of the Clairaut type and its solution, when $p = A + B\rho^{\gamma}$, is

$$f(\rho) = A + q_{max}^2 \rho/2 - B\rho^{\gamma}/(\gamma - 1) \tag{13}$$

where $q_{max}^2 = \frac{2}{\gamma - 1} a_o^2 + q_o^2$ and $a^2 = \frac{dp}{d\rho} = B\gamma\rho^{\gamma-1}$. The condition (11) is satisfied if the domain is finite because the boundary conditions in fluid dynamics are either that the boundary is a stream surface, thus $q_n = 0$, or that the velocity is prescribed; that is, $\delta\phi = 0$. For the domain which extends to infinity, since a part of the boundary S is infinite, the integral may result in a finite value, although the integrand may approach zero at infinity. This case has already been discussed in detail by Wang (reference 4) and in such cases this finite value of integral (11) should be subtracted from integral (6).

In applying the Rayleigh-Ritz method to the problem, it is more convenient to use only one variable in the variational integral, say the velocity potential ϕ, and to express the other variables u, v, and ρ in terms of ϕ by means of equations (8), (9), and (10). In so doing, equations (8), (9), and (10) are automatically satisfied and the first variation of equation (6) then leads to the fundamental differential equation (5). With equations (8), (9), (10), and (12) satisfied, it can be easily verified that the integrand of I is just the pressure p.

The Rayleigh-Ritz method consists essentially of choosing a series of functions which satisfies the boundary conditions but has undetermined parameters, such as A_{ij}, $i,j = 1, 2, \ldots$, in the functions. If the

variational integral is denoted by I, then by substituting these functions into I and setting the first variation of I equal to zero, the following set of simultaneous algebraic equations is obtained:

$$\frac{\partial I}{\partial A_{ij}} = 0 \qquad\qquad i,j = 1, 2, \ldots$$

The parameters A_{ij} can thus be determined by solving these simultaneous equations. The Rayleigh-Ritz procedure, therefore, gives a solution which satisfies the boundary conditions exactly but satisfies the differential equations only approximately.

Consider the case of a compressible-fluid flow past a circular cylinder with unit radius. The velocity potential ϕ in this case may be assumed to be

$$\phi = \phi_1 + \phi_2 \tag{14}$$

with

$$\phi_1 = U\left(r + \frac{1}{r}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta \tag{15}$$

and

$$\phi_2 = \sum_{m=1}^{M}\sum_{n=1}^{N}\left(\frac{1}{mr^m} - \frac{1}{(m + 2)r^{m+2}}\right)\left(A_{mn}\cos n\theta + B_{mn}\sin n\theta\right) \tag{16}$$

where Γ is the strength of circulation, A_{mn} and B_{mn} are undetermined parameters, and r and θ are the polar coordinates. It is evident that for the assumed value of ϕ the boundary conditions are satisfied; that is, at $r = \infty$ and $q = U$, U being the velocity of the undisturbed flow at infinity and on the surface of the body, $\partial\phi/\partial n = 0$ or the normal component of the velocity is zero, n being the inward normal direction.

The circular cylinder in the complex z-plane may be mapped into cylindrical bodies of arbitrary shape in the complex ζ-plane by the mapping function $\zeta = f(z)$. When the mapping is conformal, it was proved in reference 4 that $\phi(\zeta)$ still satisfies the boundary conditions. With ϕ as assumed in equations (15) and (16), the final variational principle (reference 4) may be written as follows:

$$I = \iint_{z\text{-plane}} \left[q_{max}^{2} - \frac{q(z)^2}{\left|\frac{d\zeta}{dz}\right|^2} \right]^{\gamma/(\gamma-1)} \left|\frac{d\zeta}{dz}\right|^2 r\,dr\,d\theta +$$

$$\frac{2\gamma}{\gamma - 1}\left(q_{max}^{2} - U^2\right)^{1/(\gamma-1)} U\pi A_{11} \tag{17}$$

where the last term is due to the fact that the integral (11) is not zero and

$$q(z)^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{r\partial\theta}\right)^2 \tag{18}$$

The final velocity in the plane of this arbitrary body is

$$q(\zeta)^2 = q(z)^2 \left|\frac{dz}{d\zeta}\right|^2 \tag{19}$$

The integral equation (17) can render itself into analytical integration only when $\gamma/(\gamma - 1)$ is an integer. For different values of $\gamma/(\gamma - 1)$, the corresponding values of γ are as follows:

$\gamma/(\gamma - 1)$	γ
$\frac{1}{2}$	-1
2	2
3	1.5
4	1.333

The isentropic value of γ for air is 1.405, which is between the values of 1.5 and 1.333. Among these values of γ, $\gamma = 1.333$ gives the closest approximation to the isentropic value. However, in this case the integrand contains an expression which is raised to the fourth power and the labor in carrying out the integration becomes excessive. In order to simplify the numerical work, $\gamma = 2$ is used in the subsequent calculation. Chaplygin and Von Kármán and Tsien used $\gamma = -1$ in their solution by the hodograph method. The justification of taking $\gamma = -1$ has also been discussed by the latter two authors. The idea is

that the velocity in the flow field does not differ too much from the undisturbed velocity and therefore only a small portion of the $p - \rho$ curve will be used. If one makes the curve with $\gamma = -1$ tangent to the curve with $\gamma = 1.405$ at the point that represents the conditions at infinity (p_o,ρ_o) and uses the curve with $\gamma = -1$, the error will not be too large. The use of $\gamma = 2$ instead of $\gamma = -1$ gives even better approximation as can be readily seen from figure 1. Although the curves with $\gamma = -1$ and $\gamma = 1.405$ may be made tangent to each other at (p_o,ρ_o), their second derivatives have opposite signs. When the curves with $\gamma = 2$ and $\gamma = 1.405$ are made tangent to each other at (p_o,ρ_o), however, a small portion of the curves is very close together, and thus a much better approximation is provided.

The isentropic pressure-density relationship is as follows:

$$p' = p_o'(\rho')^{1.405}/(\rho_o')^{1.405} \tag{20}$$

According to the previous discussion, the constants A and B in equation (3) can be determined from the condition that the curve given by equation (20) and that by equation (3) are tangent to each other at (p_o,ρ_o). This yields two conditions; namely,

$$\left.\begin{aligned} p_o &= p_o' \\ \rho_o &= \rho_o' \end{aligned}\right\} \tag{21}$$

and

$$\left(\frac{dp}{d\rho}\right)_{p_o,\rho_o} = \left(\frac{dp'}{d\rho'}\right)_{p_o,\rho_o} = a_o^2 \tag{22}$$

Thus in the case of $\gamma = 2$, the constants A and B are found to be $A = 0.2975p_o$ and $B = 0.7025p_o/\rho_o^2$ and equation (20) becomes

$$p = 0.2975p_o + \frac{0.7025p_o\rho^2}{\rho_o^2} \tag{23}$$

For $\gamma = -1$, the corresponding pressure-density relationship results in

$$p = 2.405p_o - \frac{1.405p_o\rho_o}{\rho} \tag{24}$$

For flow past the circular cylinder, the range of variation of p/p_o is approximately from 0.2 to 1.2. In this region, it can be easily seen from figure 1 that equation (23) does indeed give a very good approximation of equation (20).

FLOW PAST A CIRCULAR CYLINDER

The problem of determining the flow past a circular cylinder without circulation has been carried out by Wang and reported in reference 4. A detailed discussion has been made there for the flow at $M_o = 0.4$. Because of the symmetry of the problem, ϕ may be assumed as follows:

$$\begin{aligned}\phi = {} & U\left(r + \frac{1}{r}\right)\cos\theta + A_{11}\left(\frac{1}{r} - \frac{1}{3r^3}\right)\cos\theta + A_{13}\left(\frac{1}{r} - \frac{1}{3r^3}\right)\cos 3\theta + \\ & A_{31}\left(\frac{1}{3r^3} - \frac{1}{5r^5}\right)\cos\theta + A_{33}\left(\frac{1}{3r^3} - \frac{1}{5r^5}\right)\cos 3\theta + \\ & A_{15}\left(\frac{1}{r} - \frac{1}{3r^3}\right)\cos 5\theta + A_{51}\left(\frac{1}{5r^5} - \frac{1}{7r^7}\right)\cos\theta \end{aligned} \tag{25}$$

In expression (25), six unknown parameters are taken. The variational integral is

$$I = \int_{r=1}^{r=\infty}\int_{\theta=0}^{\theta=2\pi}\left[q_{max}^2 - \left(\frac{\partial\phi}{\partial r}\right)^2 - \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right]^2 r\,dr\,d\theta + 4\left(q_{max}^2 - U^2\right)U\pi A_{11} \tag{26}$$

where γ is taken as 2.

Substituting ϕ as given by equation (25) into equation (26) and differentiating the resulting integral with respect to the parameters, one obtains

$$\partial I/\partial A_{11} = 0 \qquad \partial I/\partial A_{13} = 0 \qquad \partial I/\partial A_{31} = 0$$

$$\partial I/\partial A_{33} = 0 \qquad \partial I/\partial A_{15} = 0 \qquad \partial I/\partial A_{51} = 0$$

Integrating these equations, there results

$$\left.\begin{aligned}
&\left(2.07407q_{max}^{2} - 7.31852U^{2}\right)A_{11} + 7.06667U^{2}A_{13} + \left(0.325926q_{max}^{2} - \right.\\
&\left.1.03704U^{2}\right)A_{31} + 0.915555U^{2}A_{33} + \left(0.114286q_{max}^{2} - 0.340136U^{2}\right)A_{51} = K_{1}\\[2ex]
&7.06667U^{2}A_{11} + \left(13.3333q_{max}^{2} - 44.2667U^{2}\right)A_{13} + 1.20000U^{2}A_{31} + \\
&\left(1.60000q_{max}^{2} - 5.97334U^{2}\right)A_{33} + 32.4444U^{2}A_{15} + 0.440816U^{2}A_{51} = K_{2}\\[2ex]
&\left(0.325926q_{max}^{2} - 1.03704U^{2}\right)A_{11} + 1.20000U^{2}A_{13} + \left(0.0900741q_{max}^{2} - \right.\\
&\left.0.207407U^{2}\right)A_{31} + 0.192254U^{2}A_{33} + \left(0.0411428q_{max}^{2} - 0.0810885U^{2}\right)A_{51} = K_{3}\\[2ex]
&0.915555U^{2}A_{11} + \left(1.60000q_{max}^{2} - 5.97334U^{2}\right)A_{13} + 0.192254U^{2}A_{31} + \\
&\left(0.277333q_{max}^{2} - 1.01181U^{2}\right)A_{33} + 4.92444U^{2}A_{15} + 0.0813062U^{2}A_{51} = K_{4}\\[2ex]
&32.4444U^{2}A_{13} + 4.92444U^{2}A_{33} + \left(35.8519q_{max}^{2} - 120.9480^{2}\right)A_{15} = K_{5}\\[2ex]
&\left(0.114286q_{max}^{2} - 0.340136U^{2}\right)A_{11} + 0.440816U^{2}A_{13} + \left(0.0411428q_{max}^{2} - \right.\\
&\left.0.0810885U^{2}\right)A_{31} + 0.0813062U^{2}A_{33} + \left(0.0218309q_{max}^{2} - \right.\\
&\left.0.0360739U^{2}\right)A_{51} = K_{6}
\end{aligned}\right\rbrace \quad (27)$$

where K_1, K_2, K_3, K_4, K_5, and K_6 are sums of nonlinear terms in A_{11}, A_{13}, A_{31}, A_{33}, A_{15}, and A_{51} and their complete expressions are given in reference 4, except that the first term in K_1 should be $4.22222U_3$ and the first term in K_3 should be $-6.66666U_3$; the corresponding values given in reference 4 are due to misprints. These equations can be solved as follows: First, assume the values of the A's to be zero in the K's. Equation (27) then becomes a system of linear simultaneous equations which can be solved by the method of solving

linear simultaneous equations suggested by Crout (reference 5). The values of A_{11}, A_{13}, A_{31}, A_{33}, A_{15}, and A_{51} so computed may be taken as the first approximation. Use the values of the A's so determined to compute the K's and solve for these parameters again. This gives the second approximation. Repeat the cycle until the desired accuracy is obtained. To make the convergence more rapid, after the third approximation, instead of following the above-outlined method rigidly, one may extrapolate the results and use the values of the A's so determined to repeat the process. This alternative method is found to be especially time saving in the computation for the cases where the Mach number is relatively high. A detailed example of the computation may be found in reference 4.

The parameters A_{ij}/a_o for flows at various undisturbed-stream Mach numbers are computed and are tabulated in table I. In order to see the convergence of the series, these parameters are computed by taking one parameter only, two parameters, three parameters, and so forth, up to six parameters. The maximum velocities at various Mach numbers are tabulated in table II for all the six cases. The convergence of the series is seen to be satisfactory except at $M_o = 0.5$ where the series is apparently divergent, and with four terms the maximum velocity is already greater than the maximum allowable velocity q_{max}. For $\gamma = 2$, $\left(\frac{q_{max}}{a_o}\right)^2 = 2 + M_o^2$. The velocity distribution over the cylinder at $M_o = 0.4$ computed by the variational method is plotted in figure 2 as are the results by the Rayleigh-Janzen method to the third approximation as computed by Imai (see reference 6). At various Mach numbers of the undisturbed flow, the maximum local Mach number is plotted in figure 3 and the local Mach number over the surface of the cylinder is plotted in figure 4. The velocities computed by the Rayleigh-Janzen and the present method show excellent agreement.

In order to study the effect on the velocity of using $\gamma = 2$, the next value closer to the isentropic one, that is, $\gamma = 1.5$, is now taken to compute the velocity. Taking only the first term in the velocity-potential series, the equation $\partial I/\partial A_{11} = 0$ is obtained as follows:

$$\left(2.11111q_{max}{}^2U^3 - 3.28889U^5\right) - \left(0.518518q_{max}{}^4 - 3.65926q_{max}{}^2U^2 + 5.92804U^4\right)A_{11} - \left(3.21005U^2 - 0.82222q_{max}{}^2\right)UA_{11}{}^2 - \left(1.05766U^2 - 0.10159q_{max}{}^2\right)A_{11}{}^3 - 0.16490UA_{11}{}^4 - 0.010564A_{11}{}^5 = 0 \qquad (28)$$

The maximum velocity with $\gamma = 1.5$ is also included in table II for comparison. It is seen that these velocities are smaller than the corresponding ones computed by taking $\gamma = 2$. However, the velocities also increase with increased number of terms in the ϕ series. Since only a finite number of terms can be taken in the numerical calculation, the velocities computed will always be smaller than the corresponding exact values. In such cases, the effect of taking $\gamma = 2$ is to increase the velocity, which compensates to some extent the effect of taking only a finite number of terms.

The pressure coefficient at a point whose pressure is p may be defined as follows:

$$C_p = \frac{p - p_o}{\frac{1}{2}\rho_o U^2} = \frac{2}{\gamma M_o^2}\left\{\left[1 + \frac{\gamma - 1}{2} M_o^2\left(1 - \frac{q^2}{U^2}\right)\right]^{\frac{\gamma}{\gamma-1}} - 1\right\} \tag{29}$$

For $\gamma = 2$,

$$C_p = \frac{1}{M_o^2}\left\{\left[1 + \frac{M_o^2}{2}\left(1 - \frac{q^2}{U^2}\right)\right]^2 - 1\right\} \tag{30}$$

The pressure coefficient is computed from the above formula and is plotted in figure 5. Results by the Rayleigh-Janzen method and the variational method compare very well, with the former yielding slightly lower results at small angles. The pressure coefficient by the Rayleigh-Janzen method was computed by taking $\gamma = 1.405$ in equation (29), which becomes upon substitution:

$$C_p = \frac{1}{0.7025M_o^2}\left\{\left[1 + 0.2025M_o^2\left(1 - \frac{q^2}{U^2}\right)\right]^{3.469} - 1\right\} \tag{31}$$

Both the variational and Rayleigh-Janzen pressure-coefficient values are lower than those obtained by the Von Kármán-Tsien method and generally higher than the Prandtl-Glauert approximation. For further comparison, the pressure-coefficient ratio $C_p/C_{p,i}$ is plotted in figure 6 where $C_{p,i}$ is the pressure coefficient of the corresponding incompressible flow.

FLOW PAST A CURVED SURFACE

In order to obtain further evidence of the applicability of the method, the compressible flow past a thin curved surface will next be considered. This example is carried out for two purposes. First, from the previous example, the variational method is seen to give very good results for flow past a thick body where the Mach number is relatively low; it is to be ascertained now whether the method will also give good results for flow past a thin body at high Mach numbers. Second, there may be some difficulty in carrying out the method when an arbitrary mapping function is used. If this is so, a study should be made to remove such difficulty.

The compressible flow past a thin bump has been carried out by Kaplan (reference 1) by the perturbation method in terms of a thickness parameter. Results are given for flow past a series of bumps of various thicknesses. Kaplan's bump may be obtained by mapping conformally a circle of unit radius in the complex z-plane into the complex ζ-plane by the mapping function

$$\zeta = z + \frac{1 - d^2}{z} + \frac{d^2}{3z^3} \tag{32}$$

Again take six parameters in ϕ as in the previous example, noting that $z = re^{i\theta}$. The equations $\partial I/\partial A_{ij} = 0$ may now be written in the following form:

$$\begin{aligned}
&{q_{max}}^2 \int_1^\infty \left(\int_0^{2\pi} \frac{\partial \phi}{\partial r} \frac{\partial}{\partial A_{ij}} \frac{\partial \phi}{\partial r}\, d\theta \right) r\, dr + {q_{max}}^2 \int_1^\infty \left(\int_0^{2\pi} \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial A_{ij}} \frac{\partial \phi}{\partial \theta}\, d\theta \right) \frac{dr}{r} - \\
&\int_1^\infty \left[\int_0^{2\pi} \lambda \left(\frac{\partial \phi}{\partial r}\right)^3 \frac{\partial}{\partial A_{ij}} \left(\frac{\partial \phi}{\partial r}\right) d\theta \right] r\, dr - \int_1^\infty \left[\int_0^{2\pi} \lambda \left(\frac{\partial \phi}{\partial r}\right)^2 \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial A_{ij}} \frac{\partial \phi}{\partial \theta}\, d\theta \right] \frac{dr}{r} - \\
&\int_1^\infty \left[\int_0^{2\pi} \lambda \left(\frac{\partial \phi}{\partial \theta}\right)^2 \frac{\partial \phi}{\partial r} \frac{\partial}{\partial A_{ij}} \left(\frac{\partial \phi}{\partial r}\right) d\theta \right] \frac{dr}{r} - \int_1^\infty \left[\int_0^{2\pi} \lambda \left(\frac{\partial \phi}{\partial \theta}\right)^3 \frac{\partial}{\partial A_{ij}} \left(\frac{\partial \phi}{\partial \theta}\right) d\theta \right] \frac{dr}{r^3} - \\
&\delta_{ij} \left({q_{max}}^2 - U^2\right) U\pi = 0
\end{aligned} \tag{33}$$

where

$$\lambda = \frac{1}{\left|\frac{d\zeta}{dz}\right|^2}$$

$$= -\frac{16d^2}{r^4}\left[\cos^2\theta - \frac{r^2}{4}\left(1 + \frac{1}{r^2}\right)^2\right]\left[\cos^2\theta + \frac{r^2}{4d^2}\left(1 - \frac{d^2}{r^2}\right)^2\right]$$

obtained from the mapping function (32). The quantity $\delta_{ij} = 1$ when $i = j = 1$ and $\delta_{ij} = 0$ for other values of i and j.

Substituting into the integrals (33) the expressions (25) and carrying out the integration first with respect to the angle θ, the resulting integrals are of the type

$$I_n = \int_0^{2\pi} \lambda f(\sin\theta \cos\theta)d\theta$$

This integration can be somewhat simplified by the use of the theory of residue. Let $Z = e^{i\theta}$ and express the integrand of I as a function of Z. Each I then consists of a sum of integrals of the two following types:

$$J_n = \oint \frac{Z^n\, dZ}{\left(Z^2 - \frac{1}{r^2}\right)\left(Z^2 + \frac{d^2}{r^2}\right)\left(Z^2 - r^2\right)\left(Z^2 + \frac{r^2}{d^2}\right)}$$

$$= \frac{d^2r^{7-n}\left\{\left[(-1)^{\frac{n-1}{2}} d^{n-1} - 1\right]r^4 - \left[(-1)^{\frac{n-1}{2}} d^{n-1} - d^4\right]\right\}}{\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)}$$

and

$$J'_n = \oint \frac{dZ}{Z^n\left(Z^2 - r^{-2}\right)\left(Z^2 + d^2r^{-2}\right)\left(Z^2 - r^2\right)\left(Z^2 + r^2d^{-2}\right)}$$

$$= \frac{d^2r^{1-n}\left\{\left[(-1)^{\frac{n+1}{2}} d^{n+5} - 1\right]r^4 - \left[(-1)^{\frac{n+1}{2}} d^{n+5} - d^4\right]\right\}}{\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)}$$

There are altogether 66 I_n integrals encountered in the calculation, and they are given in appendix A. With these expressions of I_n substituted into the integrals, the resulting integrands are rational functions of r with coefficients in d. It is observed that the final coefficients of the unknown parameters A_{ij} are sums of the following functions:

$$Y_n = \int_1^\infty \frac{r^n\,dr}{\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)}$$

where n may be either positive or negative. There are altogether 20 such Y_n expressions encountered in the calculation, and they are given in appendix B. It may be noted that terms like $\left[\log_e\left(1 - r^2\right)\right]_{r=}$ and $\left(\log_e r\right)_{r=\infty}$ appear after the integration. These terms however will be canceled by each other in the final summation. This fact may conveniently be regarded as a check to the correctness of the computati

The numerical values of d^2 may be substituted at this stage befo the final integration with respect to r is carried out; otherwise the computation will be extremely laborious. The integrands now may be tak as the products of r expressions and I_n and then the final answers can be immediately written down from the known Y_n values. There are eight r expressions involved, and they are

$$P = r + \frac{1}{r} \qquad Q = \frac{1}{r} - \frac{1}{3r^3}$$

$$R = \frac{1}{3r^3} - \frac{1}{5r^5} \qquad S = \frac{1}{5r^5} - \frac{1}{7r^7}$$

$$K = 1 - \frac{1}{r^2} \qquad L = -\frac{1}{r^2} + \frac{1}{r^4}$$

$$M = -\frac{1}{r^4} + \frac{1}{r^6} \qquad N = -\frac{1}{r^6} + \frac{1}{r^8}$$

As a numerical example, $d^2 = 0.075$ is taken, which corresponds to a bump with a maximum thickness of 5.13 percent of the chord. The shape of the bump is shown in figure 7. For $d^2 = 0.075$, the values of Y_n are calculated and included in appendix B. After dropping a common factor of 4, integrals (33), which are equivalent to

$$\partial I/\partial A_{11} = 0 \qquad \partial I/\partial A_{13} = 0 \qquad \partial I/\partial A_{31} = 0$$

$$\partial I/\partial A_{33} = 0 \qquad \partial I/\partial A_{15} = 0 \qquad \partial I/\partial A_{51} = 0$$

become

$$
\left.
\begin{aligned}
&\left(0.518519q_{max}{}^{2} - 1.193255U^{2}\right)A_{11} + 0.554586U^{2}A_{13} + \left(0.0814815q_{max}{}^{2} - \right.\\
&\left. 0.169149U^{2}\right)A_{31} + 0.0285596U^{2}A_{33} + 0.216663U^{2}A_{15} + \left(0.0285714q_{max}{}^{2} - \right.\\
&\left. 0.0546889U^{2}\right)A_{51} = K_{1}\\[2ex]
&0.554586U^{2}A_{11} + \left(3.333333q_{max}{}^{2} - 8.188034U^{2}\right)A_{13} + 0.112085U^{2}A_{31} +\\
&\left(0.400000q_{max}{}^{2} - 1.012468U^{2}\right)A_{33} + 2.347577U^{2}A_{15} + 0.0457158U^{2}A_{51} = K_{2}\\[2ex]
&\left(0.0814815q_{max}{}^{2} - 0.169149U^{2}\right)A_{11} + 0.112085U^{2}A_{13} + \left(0.0225185q_{max}{}^{2} - \right.\\
&\left. 0.0420399U^{2}\right)A_{31} + 0.00795193U^{2}A_{33} + 0.0706980U^{2}A_{15} +\\
&\left(0.0102858q_{max}{}^{2} - 0.0178331U^{2}\right)A_{51} = K_{3}\\[2ex]
&0.0285596U^{2}A_{11} + \left(0.400000q_{max}{}^{2} - 1.012468U^{2}\right)A_{13} + 0.00795193U^{2}A_{31} +\\
&\left(0.0693333q_{max}{}^{2} - 0.167616U^{2}\right)A_{33} + 0.319818U^{2}A_{15} + 0.00397346U^{2}A_{51} = K_{4}\\[2ex]
&0.216663U^{2}A_{11} + 2.347577U^{2}A_{13} + 0.0706980U^{2}A_{31} + 0.319818U^{2}A_{33} +\\
&\left(8.962963q_{max}{}^{2} - 22.05199U^{2}\right)A_{15} + 0.0356363U^{2}A_{51} = K_{5}\\[2ex]
&\left(0.0285714q_{max}{}^{2} - 0.0546889U^{2}\right)A_{11} + 0.0457158U^{2}A_{13} +\\
&\left(0.0102857q_{max}{}^{2} - 0.0178331U^{2}\right)A_{31} + 0.00397346U^{2}A_{33} +\\
&0.0356363U^{2}A_{15} + \left(0.0054577q_{max}{}^{2} - 0.00887571U^{2}\right)A_{51} = K_{6}
\end{aligned}
\right\}\quad(34)
$$

The K_n expressions are found in appendix C.

These equations (34) are solved again by the method of successive approximation as outlined in the previous section. At relatively high Mach numbers, however, the method as outlined before becomes oscillatively divergent. In such cases the average values of A_{ij} between two consecutive cycles of computations may be used for the next approximation. The process is then again convergent. Such a method was also found necessary by Wang in a different problem where another set of cubic simultaneous equations was involved (reference 7).

At $M_o = 0.9$, the method described above again failed when six parameters were used. Many schemes of computation were tried and it was not possible to calculate the values of the parameters by methods of successive approximation. It was possible, however, to calculate these values when up to four parameters were taken. This indicates that at this Mach number irrotational solution may not exist (reference 8) and shock waves have possibly occurred. It can also be seen that the velocity computed by Kaplan begins to diverge at about $M_o = 0.9$. The velocity series computed by Kaplan becomes in this case

$$\begin{aligned}\frac{q}{U} &= 1 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots \\ &= 1 + 0.176472 + 0.050675 + 0.059108 + \ldots\end{aligned}$$

where t is the maximum thickness ratio and is equal to 0.051282 in this case.

The values of A_{ij}/a_o are given in table III at $M_o = 0.5$, 0.75, 0.83, and 0.90. The velocity distribution over the surface of the bump is given in table IV and plotted in figure 7 to compare that found by the variational method with the results by Kaplan. The agreement is very good. At $M_o = 0.9$, the maximum velocities q/U with one, two, three, and four parameters are 1.1179, 1.2628, 1.2716, and 1.2630, respectively. The successive values of the velocity do not show a sign of divergence in this case. The velocities at various points on the surface of the bump are obtained as follows. For the mapping function (32), the shape of the bump in the ζ-plane is given by the following parametric equations.

$$\left.\begin{aligned}\xi &= 2\cos\theta - \frac{d^2}{3}(3\cos\theta - \cos 3\theta) \\ \eta &= \frac{d^2}{3}(3\sin\theta - \sin 3\theta)\end{aligned}\right\} \tag{35}$$

The length of chord is given by $(2\xi)_{\theta=0}$ or $\left(2 - \frac{2d^2}{3}\right)$ and the maximum thickness is given by $(2\eta)_{\theta=\pi/2}$ or $4d^2/3$. The thickness ratio t is then equal to $2d^2/(3 - d^2)$. If the unit length is taken as the semichord of the shape, equation (35) becomes

$$\left.\begin{aligned} X &= \cos\theta - \frac{t}{4}(\cos\theta - \cos 3\theta) \\ Y &= \frac{t}{4}(3\sin\theta - \sin 3\theta) \end{aligned}\right\} \tag{36}$$

For different values of X, θ may be found from equations (36). On the surface of the bump,

$$\left|\frac{d\zeta}{dz}\right| = \left\{\left[1 - (1 - d^2)\cos 2\theta - d^2\cos 4\theta\right]^2 + \left[(1 - d^2)\sin 2\theta + d^2\sin 4\theta\right]^2\right\}^{\frac{1}{2}} \tag{37}$$

The velocity is then given by the following expression:

$$\begin{aligned} \frac{q}{U} &= \frac{-\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{r=1}}{\left|\frac{d\zeta}{dz}\right|} \\ &= \frac{1}{\left|\frac{d\zeta}{dz}\right|}\left[\left(2 + \frac{0.66667A_{11}}{a_oM_o} + \frac{0.133333A_{31}}{a_oM_o} + \frac{0.057142A_{51}}{a_oM_o}\right)\sin\theta + \right. \\ &\quad \left.\left(\frac{2A_{13}}{a_oM_o} + \frac{0.4A_{33}}{a_oM_o}\right)\sin 3\theta + \frac{3.33333A_{15}}{a_oM_o}\sin 5\theta\right] \end{aligned} \tag{38}$$

For different values of X, the corresponding values of $\sin\theta$, $\sin 3\theta$, $\sin 5\theta$, and $\left|\frac{d\zeta}{dz}\right|$ are given in table V.

The pressure coefficients at various Mach numbers are given in table VI and are plotted in figure 8. The pressure coefficients computed by the variational, the perturbation, the Von Kármán-Tsien, and the Prandtl-Glauert methods are given in table VII and are plotted in figure 9 for comparison.

RESULTS AND DISCUSSION

Compressible flows past a circular cylinder and a thin curved surface have been computed by the variational method. In the case of flow past a circular cylinder, the velocities and pressure coefficients computed are compared with the results obtained by Imai who used the Rayleigh-Janzen method up to the third approximation. In the case of flow past a thin curved surface, the velocities and pressure coefficients are compared with the results obtained by Kaplan who used the perturbation method in terms of a thickness parameter up to the third approximation. The agreement of the results between the present method and the existing methods is very good. It thus shows that the variational method will give good approximate solution both for thick and thin bodies and both at low and high Mach numbers.

In carrying out the variational method, γ is taken to be 2 instead of the isentropic value which for air is 1.405. A study is made by taking $\gamma = 1.5$ in the case of flow past a circular cylinder. The results show that the effect of using a larger value of γ is to increase the maximum velocity slightly. The same conclusion may be obtained by a study of the results of the Rayleigh-Janzen method. Theoretically, if one can take a complete set of functions in the velocity-potential series, exact solution may be obtained. In actually carrying out the method, however, only a finite number of terms can be taken. Although the introduction of more terms into the solution does not involve any additional mathematical difficulties, each new term appreciably increases the computational labor. With six terms, it is observed that the maximum velocities increase with an increased number of terms. Thus the slightly higher values for maximum velocities are increments in the right direction. By using a modified pressure-density relationship instead of the isentropic one, the pressure coefficients are hardly affected by an increase in γ.

When the undisturbed velocity increases to a value at which shock waves will probably occur, the nonlinear terms of the undetermined parameters are no longer small compared with the linear terms. The method of successive approximation in solving these simultaneous equations suddenly fails. None of the many schemes tried worked when more than four parameters were taken. An investigation has been carried out to study the reason why it is so. It was found (reference 8) that the irrotational flow solution fails to exist when the Mach number is increased beyond a certain limiting value. For the cases computed, it appears that this limiting value decreases as the number of terms increases. This explains why the method of successive approximation

fails to work with more parameters while it was still possible to obtain a solution with fewer parameters.

New York University
New York, N. Y., June 17, 1949

APPENDIX A

THE I_n EXPRESSIONS

The I_n expressions encountered in the calculation of flow past a curved surface by the variational method are as follows:

$$I_1 = \int_0^{2\pi} \lambda \cos^4\theta \, d\theta$$

$$= \frac{\pi r^4}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \left[3\left(1 + d^2\right)r^8 + 4\left(1 - d^4\right)r^6 + \left(1 + d^2\right)\left(1 - 4d^2 + d^4\right)r^4 - d^4\left(1 + d^2\right)\right]$$

$$I_2 = \int_0^{2\pi} \lambda \cos^3\theta \cos 3\theta \, d\theta$$

$$= \frac{\pi r^2}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \left[\left(1 + d^2\right)r^{10} + 3\left(1 - d^4\right)r^8 + \left(1 + d^2\right)\left(3 - 4d^2 + 3d^4\right)r^6 + \left(1 - d^8\right)r^4 - 3d^4\left(1 + d^2\right)r^2 - d^4\left(1 - d^4\right)\right]$$

$$I_3 = \int_0^{2\pi} \lambda \cos^3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \left[\left(1 - d^4\right)r^{10} + 3\left(1 + d^6\right)r^8 + 3\left(1 - d^8\right)r^6 + \left(1 - 3d^4 - 3d^6 + d^{10}\right)r^4 - 3d^4\left(1 - d^4\right)r^2 - d^4\left(1 + d^6\right)\right]$$

$$I_4 = \int_0^{2\pi} \lambda \cos^2\theta \cos^2 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[2(1 + d^2)r^{12} + 2(1 - d^4)r^{10} +$$

$$(1 + d^2)(1 - 3d^2 + d^4)r^8 + 2(1 - d^8)r^6 + (1 - d^4)(1 - d^6)r^4 -$$

$$2d^4(1 - d^4)r^2 - d^4(1 + d^6)\Big]$$

$$I_5 = \int_0^{2\pi} \lambda \cos^2\theta \cos 3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2} \Big[\left(1 + d^2\right)r^{14} + 2\left(1 - d^4\right)r^{12} +$$

$$\left(1 + d^2\right)\left(1 - d^2\right)^2 r^{10} + \left(1 - d^8\right)r^8 + \left(2 - d^4 - d^6 + 2d^{10}\right)r^6 +$$

$$\left(1 - d^4\right)\left(1 + d^8\right)r^4 - 2d^4\left(1 + d^6\right)r^2 - d^4\left(1 - d^8\right)\Big]$$

$$I_6 = \int_0^{2\pi} \lambda \cos^2\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \Big[2\left(1 + d^2\right)r^{16} + 2\left(1 - d^4\right)r^{14} -$$

$$2d^2\left(1 + d^2\right)r^{12} + \left(1 + d^{10}\right)r^8 + 2\left(1 - d^{12}\right)r^6 +$$

$$\left(1 - d^4\right)\left(1 - d^{10}\right)r^4 - 2d^4\left(1 - d^8\right)r^2 - d^4\left(1 + d^{10}\right)\Big]$$

$$I_7 = \int_0^{2\pi} \lambda \cos\theta \cos^3 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2} \left[3\left(1 - d^4\right)r^{12} + 3\left(1 + d^6\right)r^{10} + \left(1 - 3d^4 - 3d^6 + d^{10}\right)r^6 + \left(1 - d^{12}\right)r^4 - d^4\left(1 + d^6\right)r^2 - d^4\left(1 - d^8\right)\right]$$

$$I_8 = \int_0^{2\pi} \lambda \cos\theta \cos^2 3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \left[\left(1 + d^2\right)r^{16} + \left(1 - d^4\right)r^{14} + \left(2 - d^2 - d^4 + 2d^6\right)r^{12} + 2\left(1 - d^8\right)r^{10} - 2d^4\left(1 + d^2\right)r^8 + \left(1 - 2d^4 + 2d^8 - d^{12}\right)r^6 + \left(1 + d^{14}\right)r^4 - d^4\left(1 - d^8\right)r^2 - d^4\left(1 + d^{10}\right)\right]$$

$$I_9 = \int_0^{2\pi} \lambda \cos\theta \cos 3\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \left[2\left(1 - d^4\right)r^{16} + 2\left(1 + d^6\right)r^{14} + \left(1 - d^8\right)r^{12} + \left(1 - 2d^4 - 2d^6 + d^{10}\right)r^{10} - d^4\left(1 - d^4\right)r^8 + \left(1 - d^4\right)\left(1 - d^{10}\right)r^6 + \left(1 - d^{16}\right)r^4 - d^4\left(1 + d^{10}\right)r^2 - d^4\left(1 - d^{12}\right)\right]$$

$$I_{10} = \int_0^{2\pi} \lambda \cos\theta \cos^3 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8} \Big[3\left(1 + d^6\right)r^{16} + 3\left(1 - d^8\right)r^{14} -$$

$$3d^4\left(1 + d^2\right)r^{12} - 3d^4\left(1 - d^4\right)r^{10} + \left(1 - d^{16}\right)r^6 + \left(1 + d^{18}\right)r^4 -$$

$$d^4\left(1 - d^{12}\right)r^2 - d^4\left(1 + d^{14}\right)\Big]$$

$$I_{11} = \int_0^{2\pi} \lambda \sin^2\theta \cos^2\theta \, d\theta$$

$$= \frac{\pi r^4}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[\left(1 + d^2\right)r^8 -$$

$$\left(1 + d^2\right)\left(1 + d^4\right)r^4 + d^4\left(1 + d^2\right)\Big]$$

$$I_{12} = \int_0^{2\pi} \lambda \sin^2\theta \cos^2 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[2\left(1 + d^2\right)r^{12} - 2\left(1 - d^4\right)r^{10} -$$

$$\left(1 + d^2\right)\left(1 + d^2 + d^4\right)r^8 + 2\left(1 - d^8\right)r^6 - \left(1 - d^4\right)\left(1 - d^6\right)r^4 -$$

$$2d^4\left(1 - d^4\right)r^2 + d^4\left(1 + d^6\right)\Big]$$

$$I_{13} = \int_0^{2\pi} \lambda \sin^2\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \Big[2\left(1 + d^2\right)r^{16} - 2\left(1 - d^4\right)r^{14} -$$

$$2d^2\left(1 + d^2\right)r^{12} - \left(1 + d^{10}\right)r^8 + 2\left(1 - d^{12}\right)r^6 - \left(1 - d^4\right)\left(1 - d^{10}\right)r^4 -$$

$$2d^4\left(1 - d^8\right)r^2 + d^4\left(1 + d^{10}\right)\Big]$$

$$I_{14} = \int_0^{2\pi} \lambda \sin^2\theta \cos\theta \cos 3\theta \, d\theta$$

$$= -\frac{\pi r^2}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[\left(1 + d^2\right)r^{10} - \left(1 - d^4\right)r^8 -$$

$$\left(1 + d^2\right)\left(1 + d^4\right)r^6 + \left(1 - d^8\right)r^4 + d^4\left(1 + d^2\right)r^2 - d^4\left(1 - d^4\right)\Big]$$

$$I_{15} = \int_0^{2\pi} \lambda \sin^2\theta \cos\theta \cos 5\theta \, d\theta$$

$$= -\frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[\left(1 - d^4\right)r^{10} - \left(1 + d^6\right)r^8 -$$

$$\left(1 - d^8\right)r^6 + \left(1 + d^4\right)\left(1 + d^6\right)r^4 + d^4\left(1 - d^4\right)r^2 - d^4\left(1 + d^6\right)\Big]$$

$$I_{16} = \int_0^{2\pi} \lambda \sin^2\theta \cos 3\theta \cos 5\theta \, d\theta$$

$$= -\frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2}\left[\left(1 + d^2\right)r^{14} - 2\left(1 - d^4\right)r^{12} + \left(1 + d^2\right)\left(1 - d^2\right)^2 r^{10} + \left(1 - d^8\right)r^8 - \left(2 + d^4 + d^6 + 2d^{10}\right)r^6 + \left(1 - d^4\right)\left(1 + d^8\right)r^4 + 2d^4\left(1 + d^6\right)r^2 - d^4\left(1 - d^8\right)\right]$$

$$I_{17} = \int_0^{2\pi} \lambda \sin\theta \sin 3\theta \cos^2\theta \, d\theta$$

$$= \frac{\pi r^2}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)}\left[\left(1 + d^2\right)r^{10} + \left(1 - d^4\right)r^8 - \left(1 + d^2\right)\left(1 + d^4\right)r^6 - \left(1 - d^8\right)r^4 + d^4\left(1 + d^2\right)r^2 + d^4\left(1 - d^4\right)\right]$$

$$I_{18} = \int_0^{2\pi} \lambda \sin\theta \sin 3\theta \cos^2 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2}\left[\left(1 - d^4\right)r^{12} - \left(1 + d^6\right)r^{10} + \left(1 + d^4\right)\left(1 + d^6\right)r^6 - \left(1 - d^{12}\right)r^4 - d^4\left(1 + d^6\right)r^2 + d^4\left(1 - d^8\right)\right]$$

$$I_{19} = \int_0^{2\pi} \lambda \sin \theta \sin 3\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \Big[2\left(1 - d^4\right)r^{16} -$$

$$2\left(1 + d^6\right)r^{14} - \left(1 - d^8\right)r^{12} + \left(1 + 2d^4 + 2d^6 + d^{10}\right)r^{10} +$$

$$d^4\left(1 - d^4\right)r^8 + \left(1 - d^4\right)\left(1 - d^{10}\right)r^6 - \left(1 - d^{16}\right)r^4 - d^4\left(1 + d^{10}\right)r^2 +$$

$$d^4\left(1 - d^{12}\right)\Big]$$

$$I_{20} = \int_0^{2\pi} \lambda \sin \theta \sin 3\theta \cos \theta \cos 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[\left(1 + d^6\right)r^8 -$$

$$\left(1 + d^4\right)\left(1 + d^6\right)r^4 + d^4\left(1 + d^6\right)\Big]$$

$$I_{21} = \int_0^{2\pi} \lambda \sin \theta \sin 3\theta \cos \theta \cos 5\theta \, d\theta$$

$$= - \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2} \Big[\left(1 + d^2\right)r^{14} -$$

$$\left(1 + d^2\right)\left(1 + d^4\right)r^{10} - \left(1 - d^8\right)r^8 + d^4\left(1 + d^2\right)r^6 +$$

$$\left(1 - d^4\right)\left(1 + d^4\right)^2 r^4 - d^4\left(1 - d^8\right)\Big]$$

$$I_{22} = \int_0^{2\pi} \lambda \sin\theta \; 3\theta \cos 3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \Big[\left(1 + d^2\right)r^{16} - \left(1 - d^4\right)r^{14} -$$

$$d^2\left(1 + d^2\right)r^{12} + \left(1 - d^{12}\right)r^6 - \left(1 + d^{14}\right)r^4 - d^4\left(1 - d^8\right)r^2 +$$

$$d^4\left(1 + d^{10}\right)\Big]$$

$$I_{23} = \int_0^{2\pi} \lambda \sin\theta \sin 5\theta \cos^2\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[\left(1 - d^4\right)r^{10} + \left(1 + d^6\right)r^8 -$$

$$\left(1 - d^8\right)r^6 - \left(1 + d^4\right)\left(1 + d^6\right)r^4 + d^4\left(1 - d^4\right)r^2 + d^4\left(1 + d^6\right)\Big]$$

$$I_{24} = \int_0^{2\pi} \lambda \sin\theta \sin 5\theta \cos^2 3\theta \, d\theta$$

$$= \frac{-\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \Big[\left(1 + d^2\right)r^{16} - \left(1 - d^4\right)r^{14} -$$

$$\left(1 + d^2\right)\left(2 - d^2 + 2d^4\right)r^{12} + 2\left(1 - d^8\right)r^{10} + 2d^4\left(1 + d^2\right)r^8 -$$

$$\left(1 - d^4\right)\left(1 + 3d^4 + d^8\right)r^6 + \left(1 + d^{14}\right)r^4 + d^4\left(1 - d^8\right)r^2 - d^4\left(1 + d^{10}\right)\Big]$$

$$I_{25} = \int_0^{2\pi} \lambda \sin\theta \sin 5\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8}\left[\left(1 + d^6\right)r^{16} - \left(1 - d^8\right)r^{14} - d^4\left(1 + d^2\right)r^{12} + d^4\left(1 - d^4\right)r^{10} + \left(1 - d^{16}\right)r^6 - \left(1 + d^{18}\right)r^4 - d^4\left(1 - d^{12}\right)r^2 + d^4\left(1 + d^{14}\right)\right]$$

$$I_{26} = \int_0^{2\pi} \lambda \sin\theta \sin 5\theta \cos\theta \cos 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2}\left[\left(1 + d^2\right)r^{14} - \left(1 + d^2\right)\left(1 + d^4\right)r^{10} + \left(1 - d^8\right)r^8 + d^4\left(1 + d^2\right)r^6 - \left(1 - d^4\right)\left(1 + d^4\right)^2 r^4 + d^4\left(1 - d^8\right)\right]$$

$$I_{27} = \int_0^{2\pi} \lambda \sin\theta \sin 5\theta \cos\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4}\left[\left(1 + d^{10}\right)r^8 - \left(1 + d^4\right)\left(1 + d^{10}\right)r^4 + d^4\left(1 + d^{10}\right)\right]$$

$$I_{28} = \int_0^{2\pi} \lambda \sin\theta \sin 5\theta \cos 3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \Big[\left(1 - d^8\right)r^{12} - \left(1 + d^{10}\right)r^{10} -$$

$$d^4\left(1 - d^4\right)r^8 + \left(1 + d^4\right)\left(1 + d^{10}\right)r^6 - \left(1 - d^{16}\right)r^4 -$$

$$d^4\left(1 + d^{10}\right)r^2 + d^4\left(1 - d^{12}\right)\Big]$$

$$I_{29} = \int_0^{2\pi} \lambda \sin^2 3\theta \cos^2\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[2\left(1 + d^2\right)r^{12} + 2\left(1 - d^4\right)r^{10} -$$

$$\left(1 + d^2\right)\left(1 + d^2 + d^4\right)r^8 - 2\left(1 - d^8\right)r^6 - \left(1 - d^4\right)\left(1 - d^6\right)r^4 +$$

$$2d^4\left(1 - d^4\right)r^2 + d^4\left(1 + d^6\right)\Big]$$

$$I_{30} = \int_0^{2\pi} \lambda \sin^2 5\theta \cos^2\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \Big[2\left(1 + d^2\right)r^{16} + 2\left(1 - d^4\right)r^{14} -$$

$$2d^2\left(1 + d^2\right)r^{12} - \left(1 + d^{10}\right)r^8 - 2\left(1 - d^{12}\right)r^6 - \left(1 - d^4\right)\left(1 - d^{10}\right)r^4 +$$

$$2d^4\left(1 - d^8\right)r^2 + d^4\left(1 + d^{10}\right)\Big]$$

$$I_{31} = \int_0^{2\pi} \lambda \sin 3\theta \sin 5\theta \cos^2\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2} \left[\left(1 + d^2\right)r^{14} + 2\left(1 - d^4\right)r^{12} + \right.$$

$$\left(1 + d^2\right)\left(1 - d^2\right)^2 r^{10} - \left(1 - d^8\right)r^8 - \left(2 + d^4 + d^6 + 2d^{10}\right)r^6 -$$

$$\left.\left(1 - d^4\right)\left(1 + d^8\right)r^4 + 2d^4\left(1 + d^6\right)r^2 + d^4\left(1 - d^8\right)\right]$$

$$I_{32} = \int_0^{2\pi} \lambda \sin^2 3\theta \cos \theta \cos 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2} \left[\left(1 - d^4\right)r^{12} + \left(1 + d^6\right)r^{10} - \right.$$

$$\left.\left(1 + d^4\right)\left(1 + d^6\right)r^6 - \left(1 - d^{12}\right)r^4 + d^4\left(1 + d^6\right)r^2 + d^4\left(1 - d^8\right)\right]$$

$$I_{33} = \int_0^{2\pi} \lambda \sin^2 5\theta \cos \theta \cos 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \left[2\left(1 - d^4\right)r^{16} + \right.$$

$$2\left(1 + d^6\right)r^{14} - \left(1 - d^8\right)r^{12} - \left(1 + 2d^4 + 2d^6 + d^{10}\right)r^{10} +$$

$$d^4\left(1 - d^4\right)r^8 - \left(1 - d^4\right)\left(1 - d^{10}\right)r^6 - \left(1 - d^{16}\right)r^4 +$$

$$\left.d^4\left(1 + d^{10}\right)r^2 + d^4\left(1 - d^{12}\right)\right]$$

$$I_{34} = \int_0^{2\pi} \lambda \sin 3\theta \sin 5\theta \cos \theta \cos 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \left[\left(1 + d^2\right)r^{16} + \left(1 - d^4\right)r^{14} - d^2\left(1 + d^2\right)r^{12} - \left(1 - d^{12}\right)r^6 - \left(1 + d^{14}\right)r^4 + d^4\left(1 - d^8\right)r^2 + d^4\left(1 + d^{10}\right)\right]$$

$$I_{35} = \int_0^{2\pi} \lambda \sin^2 3\theta \cos \theta \cos 5\theta \, d\theta$$

$$= -\frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \left[\left(1 + d^2\right)r^{16} + \left(1 - d^4\right)r^{14} - \left(1 + d^2\right)\left(2 - d^2 + 2d^4\right)r^{12} - 2\left(1 - d^8\right)r^{10} + 2d^4\left(1 + d^2\right)r^8 + \left(1 - d^4\right)\left(1 + 3d^4 + d^8\right)r^6 + \left(1 + d^{14}\right)r^4 - d^4\left(1 - d^8\right)r^2 - d^4\left(1 + d^{10}\right)\right]$$

$$I_{36} = \int_0^{2\pi} \lambda \sin^2 5\theta \cos \theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8} \left[\left(1 + d^6\right)r^{16} + \left(1 - d^8\right)r^{14} - d^4\left(1 + d^2\right)r^{12} - d^4\left(1 - d^4\right)r^{10} - \left(1 - d^{16}\right)r^6 - \left(1 + d^{18}\right)r^4 + d^4\left(1 - d^{12}\right)r^2 + d^4\left(1 + d^{14}\right)\right]$$

$$I_{37} = \int_0^{2\pi} \lambda \sin 3\theta \sin 5\theta \cos \theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \left[\left(1 - d^8\right)r^{12} + \left(1 + d^{10}\right)r^{10} - \right.$$

$$d^4\left(1 - d^4\right)r^8 - \left(1 + d^4\right)\left(1 + d^{10}\right)r^6 - \left(1 - d^{16}\right)r^4 +$$

$$\left. d^4\left(1 + d^{10}\right)r^2 + d^4\left(1 - d^{12}\right)\right]$$

$$I_{38} = \int_0^{2\pi} \lambda \sin^4\theta \, d\theta$$

$$= \frac{\pi r^4}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \left[3\left(1 + d^2\right)r^8 - 4\left(1 - d^4\right)r^6 + \right.$$

$$\left. \left(1 + d^2\right)\left(1 - 4d^2 + d^4\right)r^4 - d^4\left(1 + d^2\right)\right]$$

$$I_{39} = \int_0^{2\pi} \lambda \sin^3\theta \sin 3\theta \, d\theta$$

$$= - \frac{\pi r^2}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \left[\left(1 + d^2\right)r^{10} - 3\left(1 - d^4\right)r^8 + \right.$$

$$\left. \left(1 + d^2\right)\left(3 - 4d^2 + 3d^4\right)r^6 - \left(1 - d^8\right)r^4 - 3d^4\left(1 + d^2\right)r^2 + d^4\left(1 - d^4\right)\right]$$

$$I_{40} = \int_0^{2\pi} \lambda \sin^3\theta \sin 5\theta \, d\theta$$

$$= -\frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[\left(1 - d^4\right)r^{10} - 3\left(1 + d^6\right)r^8 +$$

$$3\left(1 - d^8\right)r^6 - \left(1 - 3d^4 - 3d^6 + d^{10}\right)r^4 - 3d^4\left(1 - d^4\right)r^2 + d^4\left(1 + d^6\right)\Big]$$

$$I_{41} = \int_0^{2\pi} \lambda \sin^2\theta \sin^2 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)} \Big[2\left(1 + d^2\right)r^{12} - 2\left(1 - d^4\right)r^{10} +$$

$$\left(1 + d^2\right)\left(1 - 3d^2 + d^4\right)r^8 - 2\left(1 - d^8\right)r^6 + \left(1 - d^4\right)\left(1 - d^6\right)r^4 +$$

$$2d^4\left(1 - d^4\right)r^2 - d^4\left(1 + d^6\right)\Big]$$

$$I_{42} = \int_0^{2\pi} \lambda \sin^2\theta \sin 3\theta \sin 5\theta \, d\theta$$

$$= -\frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2} \Big[\left(1 + d^2\right)r^{14} -$$

$$2\left(1 - d^4\right)r^{12} + \left(1 + d^2\right)\left(1 - d^2\right)^2 r^{10} - \left(1 - d^8\right)r^8 +$$

$$\left(2 - d^4 - d^6 + 2d^{10}\right)r^6 - \left(1 - d^4\right)\left(1 + d^8\right)r^4 - 2d^4\left(1 + d^6\right)r^2 +$$

$$d^4\left(1 - d^8\right)\Big]$$

$$I_{43} = \int_0^{2\pi} \lambda \sin^2\theta \sin^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \Big[2\left(1 + d^2\right)r^{16} - 2\left(1 - d^4\right)r^{14} - 2d^2\left(1 + d^2\right)r^{12} + \left(1 + d^{10}\right)r^8 - 2\left(1 - d^{12}\right)r^6 + \left(1 - d^4\right)\left(1 - d^{10}\right)r^4 + 2d^4\left(1 - d^8\right)r^2 - d^4\left(1 + d^{10}\right)\Big]$$

$$I_{44} = \int_0^{2\pi} \lambda \sin\theta \sin^3 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^2} \Big[3\left(1 - d^4\right)r^{12} - 3\left(1 + d^6\right)r^{10} - \left(1 - 3d^4 - 3d^6 + d^{10}\right)r^6 + \left(1 - d^{12}\right)r^4 + d^4\left(1 + d^6\right)r^2 - d^4\left(1 - d^8\right)\Big]$$

$$I_{45} = \int_0^{2\pi} \lambda \sin\theta \sin^2 3\theta \sin 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \Big[\left(1 + d^2\right)r^{16} - \left(1 - d^4\right)r^{14} + \left(1 + d^2\right)\left(2 - 3d^2 + 2d^4\right)r^{12} - 2\left(1 - d^8\right)r^{10} - 2d^4\left(1 + d^2\right)r^8 - \left(1 - d^4\right)\left(1 - d^4 + d^8\right)r^6 + \left(1 + d^{14}\right)r^4 + d^4\left(1 - d^8\right)r^2 - d^4\left(1 + d^{10}\right)\Big]$$

$$I_{46} = \int_0^{2\pi} \lambda \sin\theta \sin 3\theta \sin^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \Big[2\left(1 - d^4\right)r^{16} -$$

$$2\left(1 + d^6\right)r^{14} + \left(1 - d^8\right)r^{12} - \left(1 - 2d^4 - 2d^6 + d^{10}\right)r^{10} -$$

$$d^4\left(1 - d^4\right)r^8 - \left(1 - d^4\right)\left(1 - d^{10}\right)r^6 + \left(1 - d^{16}\right)r^4 + d^4\left(1 + d^{10}\right)r^2 -$$

$$d^4\left(1 - d^{12}\right)\Big]$$

$$I_{47} = \int_0^{2\pi} \lambda \sin\theta \sin^3 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8} \Big[3\left(1 + d^6\right)r^{16} - 3\left(1 - d^8\right)r^{14} -$$

$$3d^4\left(1 + d^2\right)r^{12} + 3d^4\left(1 - d^4\right)r^{10} - \left(1 - d^{16}\right)r^6 + \left(1 + d^{18}\right)r^4 +$$

$$d^4\left(1 - d^{12}\right)r^2 - d^4\left(1 + d^{14}\right)\Big]$$

$$I_{48} = \int_0^{2\pi} \lambda \sin^2 3\theta \cos^2 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \Big[\left(1 + d^2\right)r^{16} -$$

$$d^2\left(1 + d^2\right)r^{12} - \left(1 + d^{14}\right)r^4 + d^4\left(1 + d^{10}\right)\Big]$$

$$I_{49} = \int_0^{2\pi} \lambda \sin^2 3\theta \cos 3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \left[\left(1 - d^4\right)r^{16} - \left(1 + d^6\right)r^{14} + \left(1 + d^4\right)\left(1 + d^6\right)r^{10} - d^4\left(1 + d^6\right)r^6 - \left(1 - d^{16}\right)r^4 + d^4\left(1 - d^{12}\right)\right]$$

$$I_{50} = \int_0^{2\pi} \lambda \sin^2 3\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8} \left[2\left(1 + d^2\right)r^{20} - \left(1 + d^2\right)\left(1 + d^2 + d^4\right)r^{16} - 2\left(1 - d^8\right)r^{14} + d^4\left(1 + d^2\right)r^{12} + 2\left(1 - d^4\right)\left(1 + d^4\right)^2 r^{10} - 2d^4\left(1 - d^8\right)r^6 - \left(1 + d^{18}\right)r^4 + d^4\left(1 + d^{14}\right)\right]$$

$$I_{51} = \int_0^{2\pi} \lambda \sin 3\theta \sin 5\theta \cos 3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8} \left[\left(1 + d^6\right)r^{16} - d^4\left(1 + d^2\right)r^{12} - \left(1 + d^{18}\right)r^4 + d^4\left(1 + d^{14}\right)\right]$$

$$I_{52} = \int_0^{2\pi} \lambda \sin 3\theta \sin 5\theta \cos^2 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \left[\left(1 - d^4\right)r^{16} + \left(1 + d^6\right)r^{14} - \left(1 + d^4\right)\left(1 + d^6\right)r^{10} + d^4\left(1 + d^6\right)r^6 - \left(1 - d^{16}\right)r^4 + d^4\left(1 - d^{12}\right)\right]$$

$$I_{53} = \int_0^{2\pi} \lambda \sin 3\theta \sin 5\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^{10}} \left[\left(1 - d^4\right)r^{20} - \left(1 + d^{10}\right)r^{14} + \left(1 + d^4\right)\left(1 + d^{10}\right)r^{10} - d^4\left(1 + d^{10}\right)r^6 - \left(1 - d^{20}\right)r^4 + d^4\left(1 - d^{16}\right)\right]$$

$$I_{54} = \int_0^{2\pi} \lambda \sin^2 5\theta \cos^2 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8} \left[2\left(1 + d^2\right)r^{20} - \left(1 + d^2\right)\left(1 + d^2 + d^4\right)r^{16} + 2\left(1 - d^8\right)r^{14} + d^4\left(1 + d^2\right)r^{12} - 2\left(1 - d^4\right)\left(1 + d^4\right)^2 r^{10} + 2d^4\left(1 - d^8\right)r^6 - \left(1 + d^{18}\right)r^4 + d^4\left(1 + d^{14}\right)\right]$$

$$I_{55} = \int_0^{2\pi} \lambda \sin^2 5\theta \cos 3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^{10}} \left[\left(1 - d^4\right)r^{20} + \left(1 + d^{10}\right)r^{14} - \left(1 + d^4\right)\left(1 + d^{10}\right)r^{10} + d^4\left(1 + d^{10}\right)r^6 - \left(1 - d^{20}\right)r^4 + d^4\left(1 - d^{16}\right)\right]$$

$$I_{56} = \int_0^{2\pi} \lambda \sin^2 5\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^{12}} \left[\left(1 + d^2\right)r^{24} - d^2\left(1 + d^2\right)r^{20} - \left(1 + d^{22}\right)r^4 + d^4\left(1 + d^{18}\right)\right]$$

$$I_{57} = \int_0^{2\pi} \lambda \cos^4 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \left[3\left(1 + d^2\right)r^{16} - 3d^2\left(1 + d^2\right)r^{12} + 4\left(1 - d^8\right)r^{10} - 4d^4\left(1 - d^4\right)r^6 + \left(1 + d^{14}\right)r^4 - d^4\left(1 + d^{10}\right)\right]$$

$$I_{58} = \int_0^{2\pi} \lambda \cos^3 3\theta \cos 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \left[3\left(1 - d^4\right)r^{16} + \left(1 + d^6\right)r^{14} + \left(3 - d^4 - d^6 + 3d^{10}\right)r^{10} - 3d^4\left(1 + d^6\right)r^6 + \left(1 - d^{16}\right)r^4 - d^4\left(1 - d^{12}\right)\right]$$

$$I_{59} = \int_0^{2\pi} \lambda \cos^2 3\theta \cos^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8} \left[2\left(1 + d^2\right)r^{20} + \left(1 + d^2\right)\left(1 - 3d^2 + d^4\right)r^{16} + 2\left(1 - d^8\right)r^{14} - d^4\left(1 + d^2\right)r^{12} + 2\left(1 - d^4\right)\left(1 + d^8\right)r^{10} - 2d^4\left(1 - d^8\right)r^6 + \left(1 + d^{18}\right)r^4 - d^4\left(1 + d^{14}\right)\right]$$

$$I_{60} = \int_0^{2\pi} \lambda \cos 3\theta \cos^3 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^{10}} \left[3\left(1 - d^4\right)r^{20} + 3\left(1 + d^{10}\right)r^{14} + \left(1 - 3d^4 - 3d^{10} + d^{14}\right)r^{10} - d^4\left(1 + d^{10}\right)r^6 + \left(1 - d^{20}\right)r^4 - d^4\left(1 - d^{16}\right)\right]$$

$$I_{61} = \int_0^{2\pi} \lambda \cos^4 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^{12}} \left[3\left(1 + d^2\right)r^{24} - 3d^2\left(1 + d^2\right)r^{20} + 4\left(1 - d^{12}\right)r^{14} - 4d^4\left(1 - d^8\right)r^{10} + \left(1 + d^{22}\right)r^4 - d^4\left(1 + d^{18}\right)\right]$$

$$I_{62} = \int_0^{2\pi} \lambda \sin^4 3\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^4} \left[3\left(1 + d^2\right)r^{16} - 3d^2\left(1 + d^2\right)r^{12} - 4\left(1 - d^8\right)r^{10} + 4d^4\left(1 - d^4\right)r^6 + \left(1 + d^{14}\right)r^4 - d^4\left(1 + d^{10}\right)\right]$$

$$I_{63} = \int_0^{2\pi} \lambda \sin^3 3\theta \sin 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^6} \left[3\left(1 - d^4\right)r^{16} - \left(1 + d^6\right)r^{14} - \left(3 - d^4 - d^6 + 3d^{10}\right)r^{10} + 3d^4\left(1 + d^6\right)r^6 + \left(1 - d^{16}\right)r^4 - d^4\left(1 - d^{12}\right)\right]$$

$$I_{64} = \int_0^{2\pi} \lambda \sin^2 3\theta \sin^2 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^8} \left[2\left(1 + d^2\right)r^{20} + \left(1 + d^2\right)\left(1 - 3d^2 + d^4\right)r^{16} - 2\left(1 - d^8\right)r^{14} - d^4\left(1 + d^2\right)r^{12} - 2\left(1 - d^4\right)\left(1 + d^8\right)r^{10} + 2d^4\left(1 - d^8\right)r^6 + \left(1 + d^{18}\right)r^4 - d^4\left(1 + d^{14}\right)\right]$$

$$I_{65} = \int_0^{2\pi} \lambda \sin 3\theta \sin^3 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^{10}} \left[3\left(1 - d^4\right)r^{20} - \right.$$

$$3\left(1 + d^{10}\right)r^{14} - \left(1 - 3d^4 - 3d^{10} + d^{14}\right)r^{10} + d^4\left(1 + d^{10}\right)r^6 +$$

$$\left.\left(1 - d^{20}\right)r^4 - d^4\left(1 - d^{16}\right)\right]$$

$$I_{66} = \int_0^{2\pi} \lambda \sin^4 5\theta \, d\theta$$

$$= \frac{\pi}{4\left(1 + d^2\right)\left(r^4 + d^2\right)\left(r^4 - d^4\right)\left(r^4 - 1\right)r^{12}} \left[3\left(1 + d^2\right)r^{24} - \right.$$

$$3d^2\left(1 + d^2\right)r^{20} - 4\left(1 - d^{12}\right)r^{14} + 4d^4\left(1 - d^8\right)r^{10} +$$

$$\left.\left(1 + d^{22}\right)r^4 - d^4\left(1 + d^{18}\right)\right]$$

APPENDIX B

THE Y_n EXPRESSIONS

After integration the functions Y_n can be put into closed forms in terms of $\tan^{-1} d^{-1}$ and $\log_e \frac{1 - d^2}{1 + d^2}$, where $\log_e$ indicates natural logarithms. However, for small values of d, the computation eventually involves the differences of values of the same order of magnitude. It is therefore necessary to expand these functions into infinite series in terms of d. In such forms, the functions Y_n are as follows:

$$Y_1 = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 1.333333333 - 0.200000000d^2 + 0.342857142d^4 - 0.053968253d^6 + 0.202020202d^8 - 0.024975025d^{10} + 0.143589743d^{12} - 0.014379085d^{14} + 0.111455107d^{16} + \ldots\right)$$

$$Y_3 = -\frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + 1.000000000 - 0.500000000d^2 + 0.833333333d^4 - 0.083333333d^6 + 0.450000000d^8 - 0.033333333d^{10} + 0.309523809d^{12} - 0.017857142d^{14} + 0.236111111d^{16} + \ldots\right)$$

$$Y_5 = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 1.000000000 - 0.333333333d^2 + 0.533333333d^4 - 0.009523809d^6 + 0.253968253d^8 - 0.002020202d^{10} + 0.167832167d^{12} - 0.000732601d^{14} + 0.125490196d^{16} + \ldots\right)$$

$$Y_7 = \frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(-D - \log_e 2 + d^2 - 1.500000000d^4 - 0.166666667d^6 - 0.583333333d^8 - 0.050000000d^{10} - 0.366666667d^{12} - 0.023809525d^{14} - 0.267857142d^{16} + \ldots\right)$$

$$Y_9 = \frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(-\frac{D}{2} + \frac{1}{2}\log_e 2 + d^2 - 1.333333333d^4 - 0.466666667d^6 - 0.342857142d^8 - 0.079365080d^{10} - 0.202020202d^{12} - 0.0321678833d^{14} - 0.143589743d^{16} + \ldots\right)$$

$$Y_{11} = E - \frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + d^4 - 1.500000000d^6 + 0.833333333d^8 + 0.416666667d^{10} + 0.450000000d^{12} + 0.133333333d^{14} + 0.309523809d^{16} + \ldots\right)$$

$$Y_{-1} = -\frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + 1.500000000 - 0.333333333d^2 + 0.583333333d^4 - 0.116666667d^6 + 0.366666667d^8 - 0.059523809d^{10} + 0.267857142d^{12} - 0.036111111d^{14} + 0.211111111d^{16} + \ldots\right)$$

$$Y_{-3} = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 1.533333333 - 0.142857142d^2 + 0.253968253d^4 - 0.059163060d^6 + 0.167832167d^8 - 0.032478632d^{10} + 0.125490196d^{12} - 0.020546016d^{14} + 0.100250625d^{16} + \ldots\right)$$

$$Y_{-5} = -\frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + 1.833333333 - 0.250000000d^2 + 0.450000000d^4 - 0.116666667d^6 + 0.309523809d^8 - 0.067857142d^{10} + 0.236111111d^{12} - 0.044444444d^{14} + 0.190909091d^{16} + \ldots\right)$$

$$Y_{-7} = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 1.676190475 - 0.111111111d^2 + 0.202020202d^4 - 0.056721056d^6 + 0.143589743d^8 - 0.034581105d^{10} + 0.111455107d^{12} - 0.023327549d^{14} + 0.091097307d^{16} + \ldots\right)$$

$$Y_{-9} = -\frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + 2.083333333 - 0.200000000d^2 + 0.366666667d^4 - 0.109523809d^6 + 0.267857142d^8 - 0.069444444d^{10} + 0.211111111d^{12} - 0.048051949d^{14} + 0.174242424d^{16} + \ldots\right)$$

$$Y_{-11} = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 1.787301587 - 0.090909091d^2 + 0.167832167d^4 - 0.052680652d^6 + 0.125490196d^8 - 0.034532031d^{10} + 0.100250625d^{12} - 0.024430640d^{14} + 0.083478260d^{16} + \ldots\right)$$

$$Y_{-13} = -\frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + 2.283333333 - 0.166666667d^2 + 0.309523809d^4 - 0.101190475d^6 + 0.236111111d^8 - 0.068253969d^{10} + 0.190909091d^{12} - 0.049242424d^{14} + 0.160256409d^{16} + \ldots\right)$$

$$Y_{-15} = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 1.878210678 - 0.076923076d^2 + 0.143589743d^4 - 0.048567120d^6 + 0.111455107d^8 - 0.033583958d^{10} + 0.091097307d^{12} - 0.024654731d^{14} + 0.077037037d^{16} + \ldots\right)$$

$$Y_{-17} = -\frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + 2.450000000 - 0.142857142d^2 + 0.267857142d^4 - 0.093253969d^6 + 0.211111111d^8 - 0.065909091d^{10} + 0.174242424d^{12} - 0.049145298d^{14} + 0.148351647d^{16} + \ldots\right)$$

$$Y_{-19} = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 1.955133754 - 0.066666667d^2 + 0.125490196d^4 - 0.044788440d^6 + 0.100250625d^8 - 0.032273778d^{10} + 0.083478260d^{12} - 0.024405459d^{14} + 0.071519795d^{16} + \ldots\right)$$

$$Y_{-21} = -\frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + 2.592857142 - 0.125000000d^2 + 0.236111111d^4 - 0.086111111d^6 + 0.190909091d^8 - 0.063131313d^{10} + 0.160256409d^{12} - 0.048351647d^{14} + 0.138095238d^{16} + \ldots\right)$$

$$Y_{-23} = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 2.021800421 - 0.058823529d^2 + 0.111455107d^4 - 0.041427096d^6 + 0.091097307d^8 - 0.030846682d^{10} + 0.077037037d^{12} - 0.023900748d^{14} + 0.066740822d^{16} + \ldots\right)$$

$$Y_{-25} = -\frac{1}{4\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(D + \log_e 2 + 2.717857142 - 0.111111111d^2 + 0.211111111d^4 - 0.079797980d^6 + 0.174242424d^8 - 0.060256409d^{10} + 0.148351647d^{12} - 0.047186147d^{14} + 0.129166667d^{16} + \ldots\right)$$

$$Y_{-27} = -\frac{1}{2\left(1 + d^2\right)^2\left(1 - d^2\right)}\left(\frac{D}{2} - \frac{1}{2}\log_e 2 + 2.080623951 - 0.052631578d^2 + 0.100250625d^4 - 0.038465729d^6 + 0.083478260d^8 - 0.029417990d^{10} + 0.071519795d^{12} - 0.023262562d^{14} + 0.062561094d^{16} + \ldots\right)$$

where

$$D = \left[\log_e(1 - r^2)\right]_{r=1}$$

$$E = (\log_e)_{r=\infty}$$

For $d^2 = 0.075$, the values of Y_n are as follows:

$$Y_{11} = -0.163282840 - D' + E$$

$$Y_9 = 0.193584561 - D'$$

$$Y_7 = -0.146562469 - D'$$

$$Y_5 = -0.295349926 - D'$$

$$Y_3 = -0.388303682 - D'$$

$$Y_1 = -0.455432461 - D'$$

$$Y_{-1} = -0.507831136 - D'$$

$$Y_{-3} = -0.550751149 - D'$$

$$Y_{-5} = -0.587074905 - D'$$

$$Y_{-7} = -0.618549324 - D'$$

$$Y_{-9} = -0.646311118 - D'$$

$$Y_{-11} = -0.671140539 - D'$$

$$Y_{-13} = -0.693595951 - D'$$

$$Y_{-15} = -0.714090490 - D'$$

$$Y_{-17} = -0.732938314 - D'$$

$$Y_{-19} = -0.750383821 - D'$$

$$Y_{-21} = -0.766620856 - D'$$

$$Y_{-23} = -0.781805725 - D'$$

$$Y_{-25} = -0.796066296 - D'$$

$$Y_{-27} = -0.809508489 - D'$$

where $D' = 0.233873678\left[\log_e(1 - r^2)\right]_{r=1}$.

APPENDIX C

THE K_n EXPRESSIONS

The K_n expressions encountered in the calculation of flow past a curved surface by the variational method are as follows:

$$
\begin{aligned}
K_1 = {} & 0.059999539U^3 + 0.182951490UA_{11}^2 - 0.289024707UA_{11}A_{13} - 0.099215517UA_{11}A_{15} + \\
& 0.039549244UA_{11}A_{31} - 0.055145940UA_{11}A_{33} + 0.011687044UA_{11}A_{51} + 1.521974407UA_{13}^2 - \\
& 1.708453568UA_{13}A_{15} - 0.039032242UA_{13}A_{31} + 0.462318173UA_{13}A_{33} - 0.010547763UA_{13}A_{51} + \\
& 4.062796976UA_{15}^2 - 0.000252462UA_{15}A_{31} - 0.203909165UA_{15}A_{33} + 0.000978600UA_{15}A_{51} - \\
& 0.000387406UA_{31}^2 - 0.015012518UA_{31}A_{33} + 0.003802360UA_{31}A_{51} + 0.037053721UA_{33}^2 - \\
& 0.008134189UA_{33}A_{51} - 0.000749111UA_{51}^2 + 0.033220706A_{11}^3 + 0.022268910A_{11}^2A_{13} + \\
& 0.016688092A_{11}^2A_{15} + 0.023629862A_{11}^2A_{31} - 0.004191746A_{11}^2A_{33} + 0.010268355A_{11}^2A_{51} + \\
& 0.586559872A_{11}A_{13}^2 - 0.052992785A_{11}A_{13}A_{15} + 0.030572932A_{11}A_{13}A_{31} + \\
& 0.196566992A_{11}A_{13}A_{33} + 0.017816638A_{11}A_{13}A_{51} + 1.590352068A_{11}A_{15}^2 + \\
& 0.027583320A_{11}A_{15}A_{31} + 0.026348061A_{11}A_{15}A_{33} + 0.016797344A_{11}A_{15}A_{51} + \\
& 0.007609784A_{11}A_{31}^2 + 0.008118406A_{11}A_{31}A_{33} + 0.007721948A_{11}A_{31}A_{51} + \\
& 0.019404931A_{11}A_{33}^2 + 0.005037201A_{11}A_{33}A_{51} + 0.002147644A_{11}A_{51}^2 + 0.355348460A_{13}^3 + \\
& 1.998793077A_{13}^2A_{15} + 0.126029525A_{13}^2A_{31} + 0.192711803A_{13}^2A_{33} + 0.054554804A_{13}^2A_{51} + \\
& 2.546743570A_{13}A_{15}^2 + 0.093686139A_{13}A_{15}A_{31} + 0.725826526A_{13}A_{15}A_{33} + \\
& 0.066165622A_{13}A_{15}A_{51} + 0.007521368A_{13}A_{31}^2 + 0.049606014A_{13}A_{31}A_{33} + \\
& 0.008865088A_{13}A_{31}A_{51} + 0.038026476A_{13}A_{33}^2 + 0.021958499A_{13}A_{33}A_{51} + \\
& 0.002747290A_{13}A_{51}^2 + 1.182535859A_{15}^3 + 0.354628107A_{15}^2A_{31} + 0.493373000A_{15}^2A_{33} + \\
& 0.158884518A_{15}^2A_{51} + 0.006593000A_{15}A_{31}^2 + 0.026235560A_{15}A_{31}A_{33} + \\
& 0.008096717A_{15}A_{31}A_{51} + 0.068307146A_{15}A_{33}^2 + 0.017553086A_{15}A_{33}A_{51} + \\
& 0.002597463A_{15}A_{51}^2 + 0.001063060A_{31}^3 + 0.002385806A_{31}^2A_{33} + 0.001816610A_{31}^2A_{51} + \\
& 0.005491591A_{31}A_{33}^2 + 0.003005606A_{31}A_{33}A_{51} + 0.001106980A_{31}A_{51}^2 + 0.002670255A_{33}^3 + \\
& 0.002816774A_{33}^2A_{51} + 0.000987380A_{33}A_{51}^2 + 0.000237098A_{51}^3
\end{aligned}
$$

$$K_2 = -0.130908999U^3 - 0.144512354UA_{11}^2 + 3.043948814UA_{11}A_{13} - 1.708453568UA_{11}A_{15} -$$
$$0.039032242UA_{11}A_{31} + 0.462318173UA_{11}A_{33} - 0.010547763UA_{11}A_{51} + 3.21377118UA_{13}^2 +$$
$$11.857940352UA_{13}A_{15} + 0.346476064UA_{13}A_{31} + 1.031279704UA_{13}A_{33} +$$
$$0.091769946UA_{13}A_{51} + 7.731273421UA_{15}^2 - 0.349808085UA_{15}A_{31} + 1.726043683UA_{15}A_{33} -$$
$$0.146059989UA_{15}A_{51} + 0.000692852UA_{31}^2 + 0.058004535UA_{31}A_{33} + 0.002368092UA_{31}A_{51} +$$
$$0.087200038UA_{33}^2 + 0.060930600UA_{33}A_{51} + 0.001909217UA_{51}^2 + 0.007422970A_{11}^3 +$$
$$0.586559872A_{11}^2A_{13} - 0.026496392A_{11}^2A_{15} + 0.015286466A_{11}^2A_{31} +$$
$$0.098283496A_{11}^2A_{33} + 0.008908319A_{11}^2A_{51} + 1.066045380A_{11}A_{13}^2 +$$
$$3.997586154A_{11}A_{13}A_{15} + 0.252059050A_{11}A_{13}A_{31} + 0.385423606A_{11}A_{13}A_{33} +$$
$$0.109109680A_{11}A_{13}A_{51} + 2.564743570A_{11}A_{15}^2 + 0.093686139A_{11}A_{15}A_{31} +$$
$$0.725826526A_{11}A_{15}A_{33} + 0.066165622A_{11}A_{15}A_{51} + 0.007521368A_{11}A_{31}^2 +$$
$$0.049606014A_{11}A_{31}A_{33} + 0.008865088A_{11}A_{31}A_{51} + 0.038026476A_{11}A_{33}^2 +$$
$$0.021958499A_{11}A_{33}A_{51} + 0.002747290A_{11}A_{51}^2 + 3.609289260A_{13}^3 + 8.719639140A_{13}^2A_{15} +$$
$$0.208248666A_{13}^2A_{31} + 1.811741922A_{13}^2A_{33} + 0.085998486A_{13}^2A_{51} +$$
$$25.414218049A_{13}A_{15}^2 + 0.920228170A_{13}A_{15}A_{31} - 3.000980028A_{13}A_{15}A_{33} +$$
$$0.455104944A_{13}A_{15}A_{51} + 0.039746167A_{13}A_{31}^2 + 0.086959326A_{13}A_{31}A_{33} +$$
$$0.042256162A_{13}A_{31}A_{51} + 0.324078519A_{13}A_{33}^2 + 0.040162146A_{13}A_{33}A_{51} +$$
$$0.012506092A_{13}A_{51}^2 + 10.871460415A_{15}^3 + 0.504585238A_{15}^2A_{31} + 4.344047335A_{15}^2A_{33} +$$
$$0.210736557A_{15}^2A_{51} + 0.034424058A_{15}A_{31}^2 + 0.168542116A_{15}A_{31}A_{33} +$$
$$0.045551461A_{15}A_{31}A_{51} + 0.289894836A_{15}A_{33}^2 + 0.040344011A_{15}A_{33}A_{51} +$$
$$0.015645651A_{15}A_{51}^2 + 0.001033455A_{31}^3 + 0.008085122A_{31}^2A_{33} + 0.001927351A_{31}^2A_{51} +$$
$$0.009687255A_{31}A_{33}^2 + 0.009037828A_{31}A_{33}A_{51} + 0.001182158A_{31}A_{51}^2 + 0.020369802A_{33}^3 +$$
$$0.004917098A_{33}^2A_{51} + 0.002764734A_{33}A_{51}^2 + 0.000247234A_{51}^3$$

$$
\begin{aligned}
K_3 = {} & 0.009387226U^3 + 0.019774622UA_{11}^2 - 0.039032242UA_{11}A_{13} - 0.000252462UA_{11}A_{15} - \\
& 0.000774812UA_{11}A_{31} - 0.015012518UA_{11}A_{33} + 0.003802360UA_{11}A_{51} + 0.173238032UA_{13}^2 - \\
& 0.349808085UA_{13}A_{15} + 0.001385705UA_{13}A_{31} + 0.058004535UA_{13}A_{33} + 0.002368092UA_{13}A_{51} + \\
& 0.427873864UA_{15}^2 + 0.020248502UA_{15}A_{31} - 0.035947203UA_{15}A_{33} + 0.007519578UA_{15}A_{51} - \\
& 0.002531840UA_{31}^2 - 0.004444278UA_{31}A_{33} - 0.000566987UA_{31}A_{51} + 0.003749007UA_{33}^2 - \\
& 0.002090749UA_{33}A_{51} - 0.001078941UA_{51}^2 + 0.007876620A_{11}^3 + 0.015286466A_{11}^2A_{13} + \\
& 0.013791660A_{11}^2A_{15} + 0.007609784A_{11}^2A_{31} + 0.004059203A_{11}^2A_{33} + 0.003860974A_{11}^2A_{51} + \\
& 0.126029525A_{11}A_{13}^2 + 0.093686139A_{11}A_{13}A_{15} + 0.015042736A_{11}A_{13}A_{31} + \\
& 0.049606014A_{11}A_{13}A_{33} + 0.008865088A_{11}A_{13}A_{51} + 0.354628107A_{11}A_{15}^2 + \\
& 0.013186000A_{11}A_{15}A_{31} + 0.026235560A_{11}A_{15}A_{33} + 0.008096717A_{11}A_{15}A_{51} + \\
& 0.003189180A_{11}A_{31}^2 + 0.004771612A_{11}A_{31}A_{33} + 0.003633220A_{11}A_{31}A_{51} + \\
& 0.005491591A_{11}A_{33}^2 + 0.003005606A_{11}A_{33}A_{51} + 0.001106980A_{11}A_{51}^2 + 0.069416222A_{13}^3 + \\
& 0.460114085A_{13}^2A_{15} + 0.039746167A_{13}^2A_{31} + 0.043479663A_{13}^2A_{33} + 0.021128081A_{13}^2A_{51} + \\
& 0.504585238A_{13}A_{15}^2 + 0.068848115A_{13}A_{15}A_{31} + 0.168542116A_{13}A_{15}A_{33} + \\
& 0.045551461A_{13}A_{15}A_{51} + 0.003100365A_{13}A_{31}^2 + 0.016170245A_{13}A_{31}A_{33} + \\
& 0.003354703A_{13}A_{31}A_{51} + 0.009687255A_{13}A_{33}^2 + 0.009037828A_{13}A_{33}A_{51} + \\
& 0.001182158A_{13}A_{51}^2 + 0.230447196A_{15}^3 + 0.120600559A_{15}^2A_{31} - 0.260365077A_{15}^2A_{33} + \\
& 0.067340386A_{15}^2A_{51} + 0.002708871A_{15}A_{31}^2 + 0.015381835A_{15}A_{31}A_{33} + \\
& 0.003297761A_{15}A_{31}A_{51} + 0.017641290A_{15}A_{33}^2 + 0.010074415A_{15}A_{33}A_{51} + \\
& 0.001024491A_{15}A_{51}^2 + 0.000551450A_{31}^3 + 0.001323410A_{31}^2A_{33} + 0.001028092A_{31}^2A_{51} + \\
& 0.002235226A_{31}A_{33}^2 + 0.001706461A_{31}A_{33}A_{51} + 0.000672957A_{31}A_{51}^2 + 0.000845763A_{33}^3 + \\
& 0.001341681A_{33}^2A_{51} + 0.000569815A_{33}A_{51}^2 + 0.000153120A_{51}^3
\end{aligned}
$$

$$
\begin{aligned}
K_4 = {} & -0.016023195U^3 - 0.027572970UA_{11}^2 + 0.462318173UA_{11}A_{13} - 0.203909165UA_{11}A_{15} - \\
& 0.015012518UA_{11}A_{31} + 0.074107442UA_{11}A_{33} - 0.008134189UA_{11}A_{51} + 0.515713852UA_{13}^2 + \\
& 1.726043683UA_{13}A_{15} + 0.058004535UA_{13}A_{31} + 0.174400076UA_{13}A_{33} + 0.060930600UA_{13}A_{51} + \\
& 1.217327709UA_{15}^2 - 0.035947203UA_{15}A_{31} + 0.235754137UA_{15}A_{33} - 0.021220641UA_{15}A_{51} - \\
& 0.002222139UA_{31}^2 + 0.007498014UA_{31}A_{33} - 0.002090749UA_{31}A_{51} + 0.014419153UA_{33}^2 + \\
& 0.001269098UA_{33}A_{51} - 0.000509384UA_{51}^2 - 0.001397249A_{11}^3 + 0.098283496A_{11}^2A_{13} + \\
& 0.013174031A_{11}^2A_{15} + 0.004059203A_{11}^2A_{31} + 0.019404931A_{11}^2A_{33} + 0.002518601A_{11}^2A_{51} + \\
& 0.192711803A_{11}A_{13}^2 + 0.725826526A_{11}A_{13}A_{15} + 0.049606014A_{11}A_{13}A_{31} + \\
& 0.076052952A_{11}A_{13}A_{33} + 0.021958499A_{11}A_{13}A_{51} + 0.493373000A_{11}A_{15}^2 + \\
& 0.026235560A_{11}A_{15}A_{31} + 0.136614292A_{11}A_{15}A_{33} + 0.017553086A_{11}A_{15}A_{51} + \\
& 0.002385806A_{11}A_{31}^2 + 0.010983182A_{11}A_{31}A_{33} + 0.003005606A_{11}A_{31}A_{51} + \\
& 0.008010765A_{11}A_{33}^2 + 0.005633548A_{11}A_{33}A_{51} + 0.000987380A_{11}A_{51}^2 + 0.603793974A_{13}^3 - \\
& 1.500490014A_{13}^2A_{15} + 0.043479663A_{13}^2A_{31} + 0.324078519A_{13}^2A_{33} + 0.020081073A_{13}^2A_{51} + \\
& 4.3440473555A_{13}A_{15}^2 + 0.168542116A_{13}A_{15}A_{31} + 0.5797789672A_{13}A_{15}A_{33} + \\
& 0.040344011A_{13}A_{15}A_{51} + 0.008085122A_{13}A_{31}^2 + 0.019374510A_{13}A_{31}A_{33} + \\
& 0.009037828A_{13}A_{31}A_{51} + 0.061109406A_{13}A_{33}^2 + 0.009834196A_{13}A_{33}A_{51} + \\
& 0.002764734A_{13}A_{51}^2 + 1.9323938855A_{15}^3 - 0.260365093A_{15}^2A_{31} + 0.806221543A_{15}^2A_{33} + \\
& 0.060527562A_{15}^2A_{51} + 0.007690918A_{15}A_{31}^2 + 0.035282580A_{15}A_{31}A_{33} + \\
& 0.010074415A_{15}A_{31}A_{51} + 0.056866015A_{15}A_{33}^2 + 0.018945942A_{15}A_{33}A_{51} + \\
& 0.003407649A_{15}A_{51}^2 + 0.000441137A_{31}^3 + 0.002235226A_{31}^2A_{33} + 0.000853231A_{31}^2A_{51} + \\
& 0.002537289A_{31}A_{33}^2 + 0.002683362A_{31}A_{33}A_{51} + 0.000569815A_{31}A_{51}^2 + 0.004115250A_{33}^3 + \\
& 0.001435584A_{33}^2A_{51} + 0.000864839A_{33}A_{51}^2 + 0.000129993A_{51}^3
\end{aligned}
$$

$$
\begin{aligned}
K_5 = {} & 0.009241431U^3 - 0.049607758UA_{11}^2 - 1.708453568UA_{11}A_{13} + 8.125593952UA_{11}A_{15} - \\
& 0.000252462UA_{11}A_{31} - 0.203909165UA_{11}A_{33} + 0.000978600UA_{11}A_{51} + 5.928619918UA_{13}^2 + \\
& 15.462546842UA_{13}A_{15} - 0.349808085UA_{13}A_{31} + 1.726043683UA_{13}A_{33} - \\
& 0.146059989UA_{13}A_{51} + 10.786166048UA_{15}^2 + 0.855747728UA_{15}A_{31} + 2.434655418UA_{15}A_{33} + \\
& 0.190818970UA_{15}A_{51} + 0.010124251UA_{31}^2 - 0.035947203UA_{31}A_{33} + 0.007519578UA_{31}A_{51} + \\
& 0.117877054UA_{33}^2 - 0.021220641UA_{33}A_{51} + 0.003144981UA_{51}^2 + 0.005562697A_{11}^3 - \\
& 0.026496392A_{11}^2A_{13} + 1.590352068A_{11}^2A_{15} + 0.013791660A_{11}^2A_{31} + 0.013174031A_{11}^2A_{33} + \\
& 0.008398672A_{11}^2A_{51} + 1.998793077A_{11}A_{13}^2 + 5.129487140A_{11}A_{13}A_{15} + \\
& 0.093686139A_{11}A_{13}A_{31} + 0.725826526A_{11}A_{13}A_{33} + 0.066165622A_{11}A_{13}A_{51} + \\
& 3.547607577A_{11}A_{15}^2 + 0.709256214A_{11}A_{15}A_{31} + 0.986746000A_{11}A_{15}A_{33} + \\
& 0.317769036A_{11}A_{51}A_{51} + 0.006593000A_{11}A_{31}^2 + 0.026235560A_{11}A_{31}A_{33} + \\
& 0.008096717A_{11}A_{31}A_{51} + 0.068307146A_{11}A_{33}^2 + 0.017553086A_{11}A_{33}A_{51} + \\
& 0.002597463A_{11}A_{51}^2 + 2.906546380A_{13}^3 + 25.414218049A_{13}^2A_{15} + 0.460114085A_{13}^2A_{31} - \\
& 1.500490014A_{13}^2A_{33} + 0.227552472A_{13}^2A_{51} + 32.614381245A_{13}A_{15}^2 + \\
& 1.009170476A_{13}A_{15}A_{31} + 8.688094670A_{13}A_{15}A_{33} + 0.421473114A_{13}A_{15}A_{51} + \\
& 0.034424058A_{13}A_{31}^2 + 0.168542116A_{13}A_{31}A_{33} + 0.045551461A_{13}A_{31}A_{51} + \\
& 0.289894836A_{13}A_{33}^2 + 0.040344011A_{13}A_{33}A_{51} + 0.015645650A_{13}A_{51}^2 + \\
& 36.413233786A_{15}^3 + 0.691341588A_{15}^2A_{31} + 5.797181565A_{15}^2A_{33} + 0.287053533A_{15}^2A_{51} + \\
& 0.120600559A_{15}A_{31}^2 - 0.520730154A_{15}A_{31}A_{33} + 0.134680772A_{15}A_{31}A_{51} + \\
& 0.806221543A_{15}A_{33}^2 + 0.121055124A_{15}A_{33}A_{51} + 0.041784273A_{15}A_{51}^2 + 0.000902957A_{31}^3 + \\
& 0.007690918A_{31}^2A_{33} + 0.001648881A_{31}^2A_{51} + 0.017641290A_{31}A_{33}^2 + \\
& 0.010074415A_{31}A_{33}A_{51} + 0.001024491A_{31}A_{51}^2 + 0.018955338A_{33}^3 + 0.009472971A_{33}^2A_{51} + \\
& 0.003407649A_{33}A_{51}^2 + 0.000214347A_{51}^3
\end{aligned}
$$

$$
\begin{aligned}
K_6 = {} & 0.003281673U^3 + 0.005843522UA_{11}^2 - 0.010547763UA_{11}A_{13} + 0.000978600UA_{11}A_{15} + \\
& 0.003802360UA_{11}A_{31} - 0.008134189UA_{11}A_{33} - 0.001498222UA_{11}A_{51} + 0.045884973UA_{13}^2 - \\
& 0.046059989UA_{13}A_{15} + 0.002368092UA_{13}A_{31} + 0.060930600UA_{13}A_{33} + 0.003818433UA_{13}A_{51} + \\
& 0.095409485UA_{15}^2 + 0.007519578UA_{15}A_{31} - 0.021220641UA_{15}A_{33} + 0.006289961UA_{15}A_{51} - \\
& 0.000283494UA_{31}^2 - 0.002090749UA_{31}A_{33} - 0.002157882UA_{31}A_{51} + 0.000634549UA_{33}^2 - \\
& 0.001018767UA_{33}A_{51} - 0.000752348UA_{51}^2 + 0.003422785A_{11} + 0.008908319A_{11}^2A_{13} + \\
& 0.008398672A_{11}^2A_{15} + 0.003860974A_{11}^2A_{31} + 0.002518601A_{11}^2A_{33} + 0.002147644A_{11}^2A_{51} + \\
& 0.054554840A_{11}A_{13}^2 + 0.066165622A_{11}A_{13}A_{15} + 0.008865088A_{11}A_{13}A_{31} + \\
& 0.021958499A_{11}A_{13}A_{33} + 0.005494580A_{11}A_{13}A_{51} + 0.158884518A_{11}A_{15}^2 + \\
& 0.008096717A_{11}A_{15}A_{31} + 0.017553086A_{11}A_{15}A_{33} + 0.005194926A_{11}A_{15}A_{51} + \\
& 0.001816610A_{11}A_{31}^2 + 0.003005606A_{11}A_{31}A_{33} + 0.002213960A_{11}A_{31}A_{51} + \\
& 0.002816774A_{11}A_{33}^2 + 0.001974760A_{11}A_{33}A_{51} + 0.000711294A_{11}A_{51}^2 + 0.028666162A_{13}^3 + \\
& 0.227552472A_{13}^2A_{15} + 0.021128081A_{13}^2A_{31} + 0.020081073A_{13}^2A_{33} + 0.012506092A_{13}^2A_{51} + \\
& 0.210736557A_{13}A_{15}^2 + 0.045551461A_{13}A_{15}A_{31} + 0.040344011A_{13}A_{15}A_{33} + \\
& 0.031291301A_{13}A_{15}A_{51} + 0.001927352A_{13}A_{31}^2 + 0.009037828A_{13}A_{31}A_{33} + \\
& 0.002364316A_{13}A_{31}A_{51} + 0.004917098A_{13}A_{33}^2 + 0.005529468A_{13}A_{33}A_{51} + \\
& 0.000741701A_{13}A_{51}^2 + 0.095684511A_{15}^3 + 0.067340386A_{15}^2A_{31} + 0.060527562A_{15}^2A_{33} + \\
& 0.041784273A_{15}^2A_{51} + 0.001648881A_{15}A_{31}^2 + 0.010074415A_{15}A_{31}A_{33} + \\
& 0.002048982A_{15}A_{31}A_{51} + 0.009472971A_{15}A_{33}^2 + 0.006815299A_{15}A_{33}A_{51} + \\
& 0.000643042A_{15}A_{51}^2 + 0.000342697A_{31}^3 + 0.000853231A_{31}^2A_{33} + 0.000672957A_{31}^2A_{51} + \\
& 0.001341681A_{31}A_{33}^2 + 0.001139630A_{31}A_{33}A_{51} + 0.000459360A_{31}A_{51}^2 + \\
& 0.000478528A_{33}^3 + 0.000864839A_{33}^2A_{51} + 0.000389980A_{33}A_{51}^2 + 0.000108272A_{51}^3
\end{aligned}
$$

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TABLE I

PARAMETERS A_{ij}/a_o FOR FLOW PAST A CIRCULAR CYLINDER

AT VARIOUS UNDISTURBED-STREAM MACH NUMBERS

$[\gamma = 2]$

M_o	A_{11}/a_o	A_{13}/a_o	A_{31}/a_o	A_{33}/a_o	A_{15}/a_o	A_{51}/a_o
0.1	0.001031					
	.001035	-0.0002557				
	.001103	-.0002557	-0.0004301			
	.001103	-.0002541	-.0004301	-0.00001330		
	.001103	-.0002542	-.0004301	-.00001331	0.000001180	
	.001105	-.0002541	-.0004378	-.00001337	.000001180	0.00001300
0.2	0.008605					
	.008747	-0.002199				
	.009347	-.002198	-0.003790			
	.009350	-.002140	-.003780	-0.0004862		
	.009346	-.002141	-.003780	-.0004979	0.00004361	
	.009568	-.002135	-.004123	-.0004981	.00004361	0.0005734
0.3	0.03114					
	.03278	-0.008506				
	.03520	-.008503	-0.01514			
	.03518	-.007949	-.01491	-0.004733		
	.03519	-.007973	-.01489	-.005045	0.0004367	
	.03540	-.007972	-.01803	-.005051	.0004367	0.005229
0.4	0.08307					
	.09430	-0.02580				
	.1026	-.02581	-0.04693			
	.1018	-.02225	-.04381	-0.03156		
	.1025	-.02265	-.04297	-.03722	0.003155	
	.1038	-.02264	-.06157	-.03727	.003156	0.03036
0.5	0.1979					
	.3242	-0.09897				
	.3541	-.09980	-0.1657			
	.3431	-.06491	.05611	-0.4292		

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TABLE II

MAXIMUM VELOCITIES q/U OVER A CIRCULAR CYLINDER COMPUTED BY VARIATIONAL AND RAYLEIGH-JANZEN METHODS AT VARIOUS UNDISTURBED-STREAM MACH NUMBERS

$\frac{q}{U}$ / M_o	Variational method							Rayleigh-Janzen method (third approximation)	
	$\gamma = 2$						$\gamma=1.5$		
	One term	Two terms	Three terms	Four terms	Five terms	Six terms	One term	$\gamma = 1.405$	$\gamma = 2$
0.1	2.0069	2.0120	2.0119	2.0119	2.0120	2.0120	2.0069	2.0119	2.0120
.2	2.0287	2.05115	2.0506	2.0510	2.0518	2.0524	2.0284	2.0513	2.0520
.3	2.0692	2.1296	2.1282	2.1309	2.1363	2.1364	2.0681	2.1314	2.1360
.4	2.1385	2.2862	2.2844	2.2979	2.3333	2.3336	2.1040	2.2836	2.3027
.5	[a]2.2639	[a]2.8281	[a]2.8271	[a]3.0754	------	------	2.2494	2.5707	2.6312

[a] $q_{max}/U = 3.0$.

TABLE III

PARAMETERS A_{ij}/a_o FOR FLOW PAST A KAPLAN BUMP AT VARIOUS

UNDISTURBED-STREAM MACH NUMBERS

$[\gamma = 2]$

M_o	A_{11}/a_o	A_{13}/a_o	A_{31}/a_o	A_{33}/a_o	A_{15}/a_o	A_{51}/a_o
0.5	0.009459	-0.003325	-0.001799	0.000558	0.000180	0.000156
0.75	0.05289	-0.02119	-0.01607	0.01662	0.02490	-0.01327
0.83	0.10501	-0.04533	-0.02624	0.04201	0.00799	-0.07141
0.90	0.092073 .20799 .2554 .2736	-0.081943 -.08626 -.1064	-0.1917 -.2256	0.1558		

TABLE IV

VELOCITY DISTRIBUTION q/U OVER SURFACE OF A KAPLAN BUMP COMPUTED BY VARIATIONAL AND PERTURBATION METHODS AT VARIOUS UNDISTURBED-STREAM MACH NUMBERS

$[t = 0.051282]$

x \ $\frac{q}{U}$	$M_o = 0$	Variational method ($\gamma = 2$)				Perturbation method ($\gamma = 1.405$)			
		$M_o = 0.50$	$M_o = 0.75$	$M_o = 0.83$	$M_o = 0.90$ (1)	$M_o = 0.50$	$M_o = 0.75$	$M_o = 0.83$	$M_o = 0.90$
0	1.0811	1.0952	1.1361	1.1872	1.2630	1.0953	1.1353	1.1743	1.2859
.1	1.0789	1.0926	1.1319	1.1803	1.2564	1.0928	1.1313	1.1681	1.2723
.2	1.0729	1.0854	1.1200	1.1612	1.2375	1.0853	1.1196	1.1512	1.2348
.3	1.0634	1.0739	1.1014	1.1314	1.2070	1.0734	1.1011	1.245	1.1793
.4	1.0493	1.0573	1.0760	1.0927	1.1646	1.0574	1.0771	1.0909	1.1158
.5	1.0333	1.0384	1.0475	1.0507	1.1139	1.0383	1.0493	1.0538	1.0553
.6	1.0149	1.0168	1.0169	1.0086	1.0554	1.0167	1.0193	1.0167	1.0060
.7	.9949	.9936	.9861	.9705	.9909	.9936	.9883	.9814	.9716
.8	.9738	.9695	.9569	.9401	.9218	.9696	.9583	.9488	.9449
.9	.9522	.9450	.9310	.9209	.8497	.9451	.9282	.9173	.9016
.975	.9357	.9264	.9146	.9180	.7876	.9270	.9050	.8918	.8329
1.0	-------	-------	-------	-------	-------	.9204	.8969	.8821	.7977

[1]Four parameters of the series taken.

TABLE V

VALUES OF SIN θ, SIN 3θ, SIN 5θ, AND $\left|\frac{d\zeta}{dz}\right|$

FOR VARIOUS VALUES OF X

X	sin θ	sin 3θ	sin 5θ	$\left\|\frac{d\zeta}{dz}\right\|$
0	1.00	-1.000	1.000	1.8500
.1	.9944	-.950	.8628	1.8434
.2	.9777	-.8053	.4909	1.8226
.3	.9492	-.5732	-.0300	1.7953
.4	.9086	-.2746	-.5518	1.7318
.5	.8544	.0684	-.9172	1.6537
.6	.7850	.4202	-.9794	1.5469
.7	.6963	.7385	-.6513	1.3997
.8	.5807	.9589	.0435	1.1926
.9	.4186	.9626	.8326	.87926
.975	.2119	.6156	.9609	.45294
1.00	0	0	0	0

TABLE VI

PRESSURE COEFFICIENTS COMPUTED BY VARIATIONAL METHOD

FOR VARIOUS UNDISTURBED-STREAM MACH NUMBERS AT

VARIOUS VALUES OF X ON BUMP

C_p / X	$M_o = 0$	$M_o = 0.5$	$M_o = 0.75$	$M_o = 0.83$
0	-0.168777	-0.196976	-0.278837	-0.380572
.1	-.164025	-.191428	-.270078	-.366493
.2	-.151114	-.176112	-.245300	-.327481
.3	-.130820	-.151796	-.206697	-.266558
.4	-.101030	-.117012	-.148537	-.187512
.5	-.067709	-.077888	-.095925	-.102108
.6	-.030022	-.033808	-.033924	-.017223
.7	.010174	.012772	.027712	.058712
.8	.051714	.060936	.085342	.118537
.9	.093315	.107692	.135733	.155920
.975	.124466	.143040	.167266	.161538

TABLE VII

PRESSURE COEFFICIENTS COMPUTED BY VARIOUS METHODS

FOR $M_o = 0.83$ AT VARIOUS VALUES OF X ON BUMP

X \ C_p	Prandtl-Glauert method	Von Kármán-Tsien method	Perturbation method	Variational method
0	-0.30245	-0.324138	-0.35564	-0.380572
.1	-.294033	-.314480	-.34310	-.366493
.2	-.270905	-.288167	-.30771	-.327481
.3	-.234545	-.247375	-.25271	-.266558
.4	-.181081	-.188634	-.18424	-.187512
.5	-.121378	-.124726	-.10861	-.102108
.6	-.053787	-.054434	-.03270	-.017223
.7	.018234	.018160	.03827	.058712
.8	.092715	.090853	.10222	.118537
.9	.167303	.161335	.16275	.155920
.975	.223151	.212658	.21232	.161538

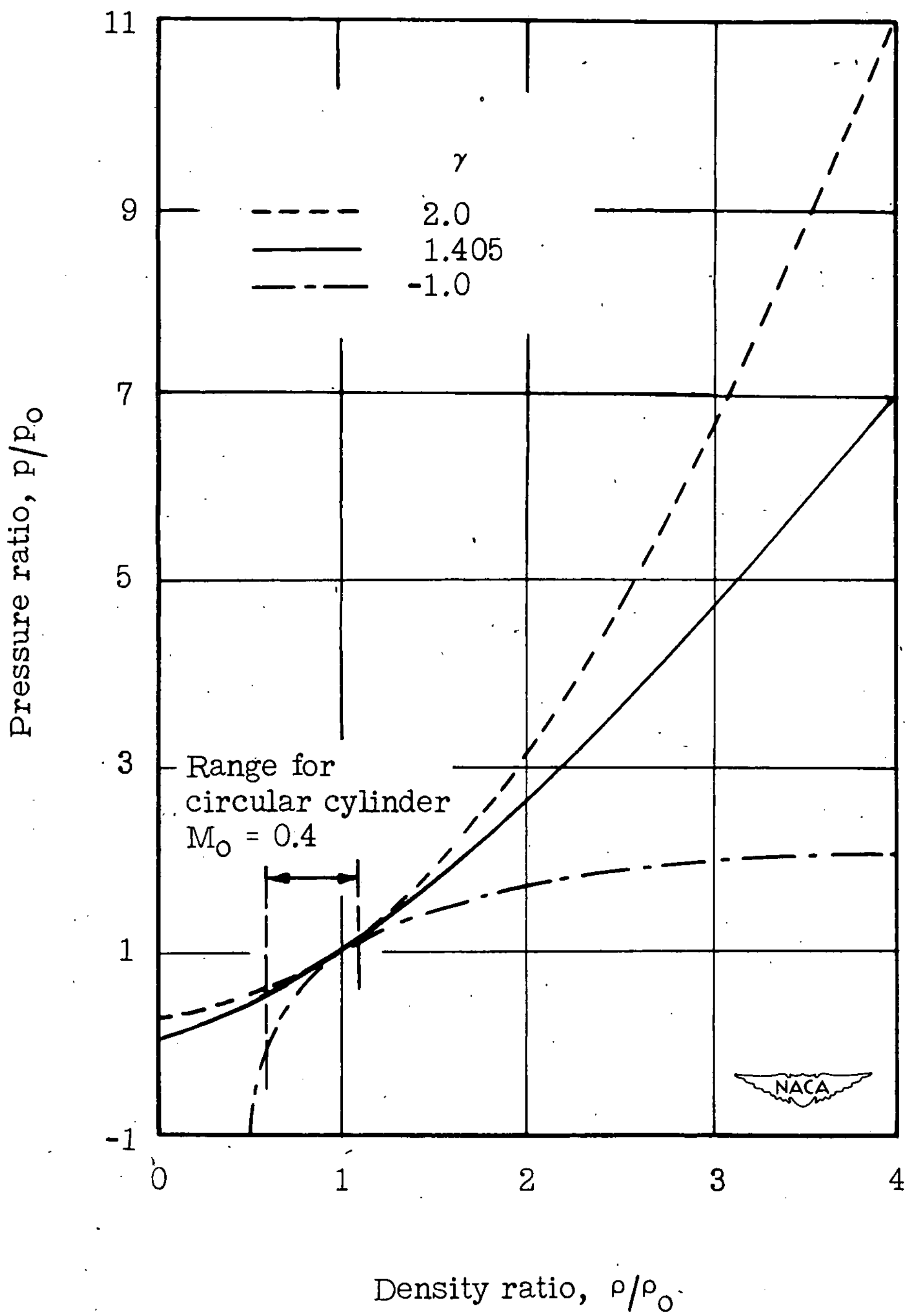

Figure 1.- Plot of pressure ratio against density ratio for various values of γ for circular cylinder.

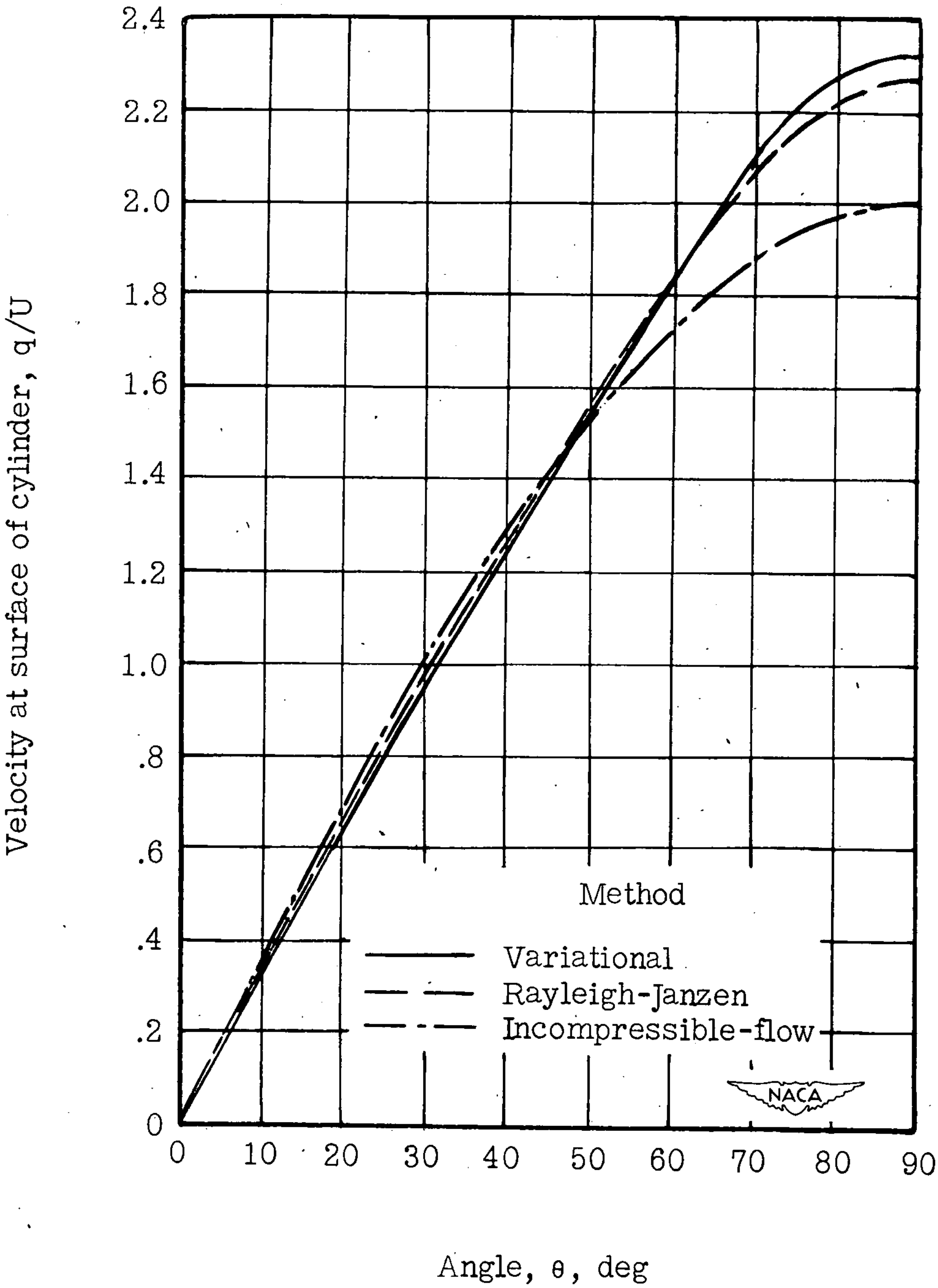

Figure 2.- Velocity distribution over a circular cylinder at $M_0 = 0.4$ computed by variational, Rayleigh-Janzen, and incompressible-flow methods.

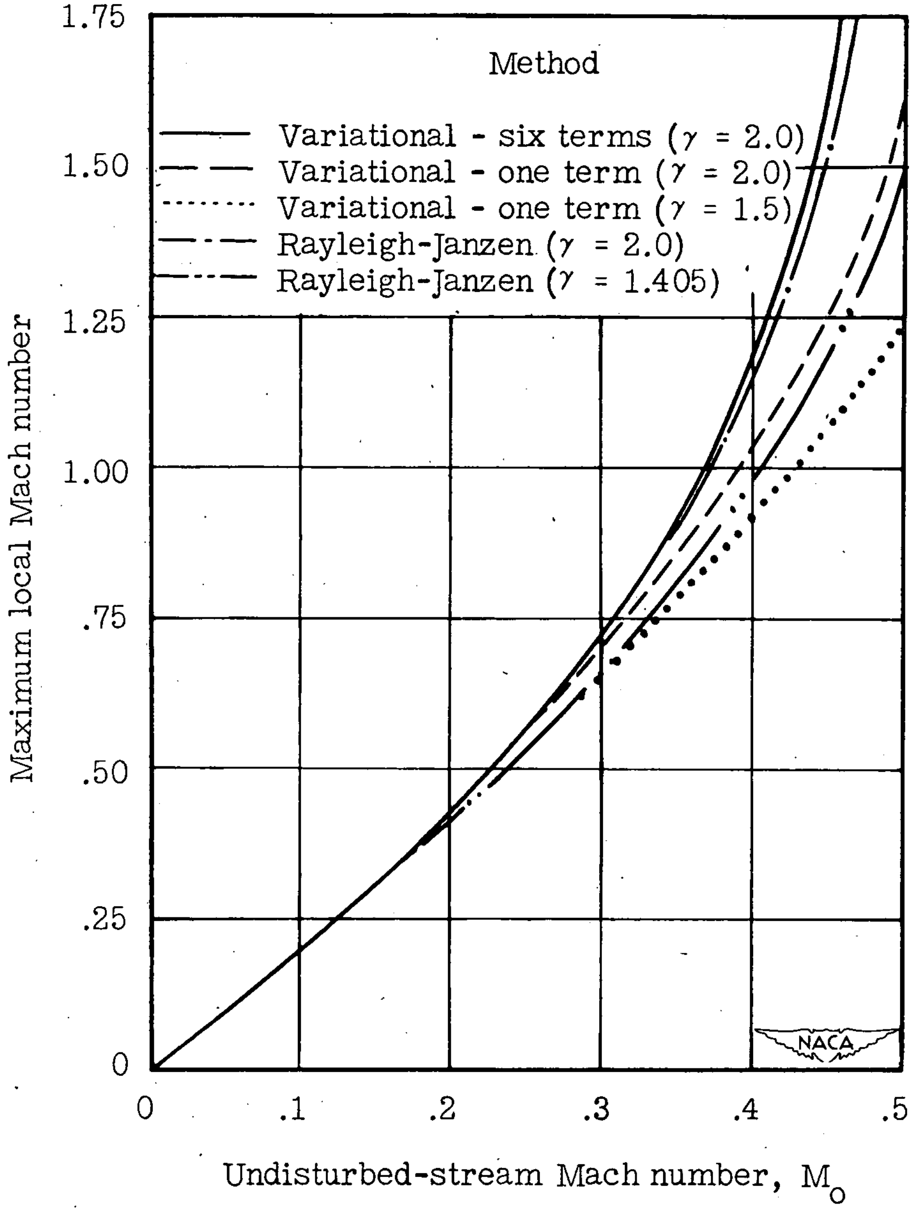

Figure 3.- Maximum local Mach number at various undisturbed-stream Mach numbers computed by variational and Rayleigh-Janzen methods.

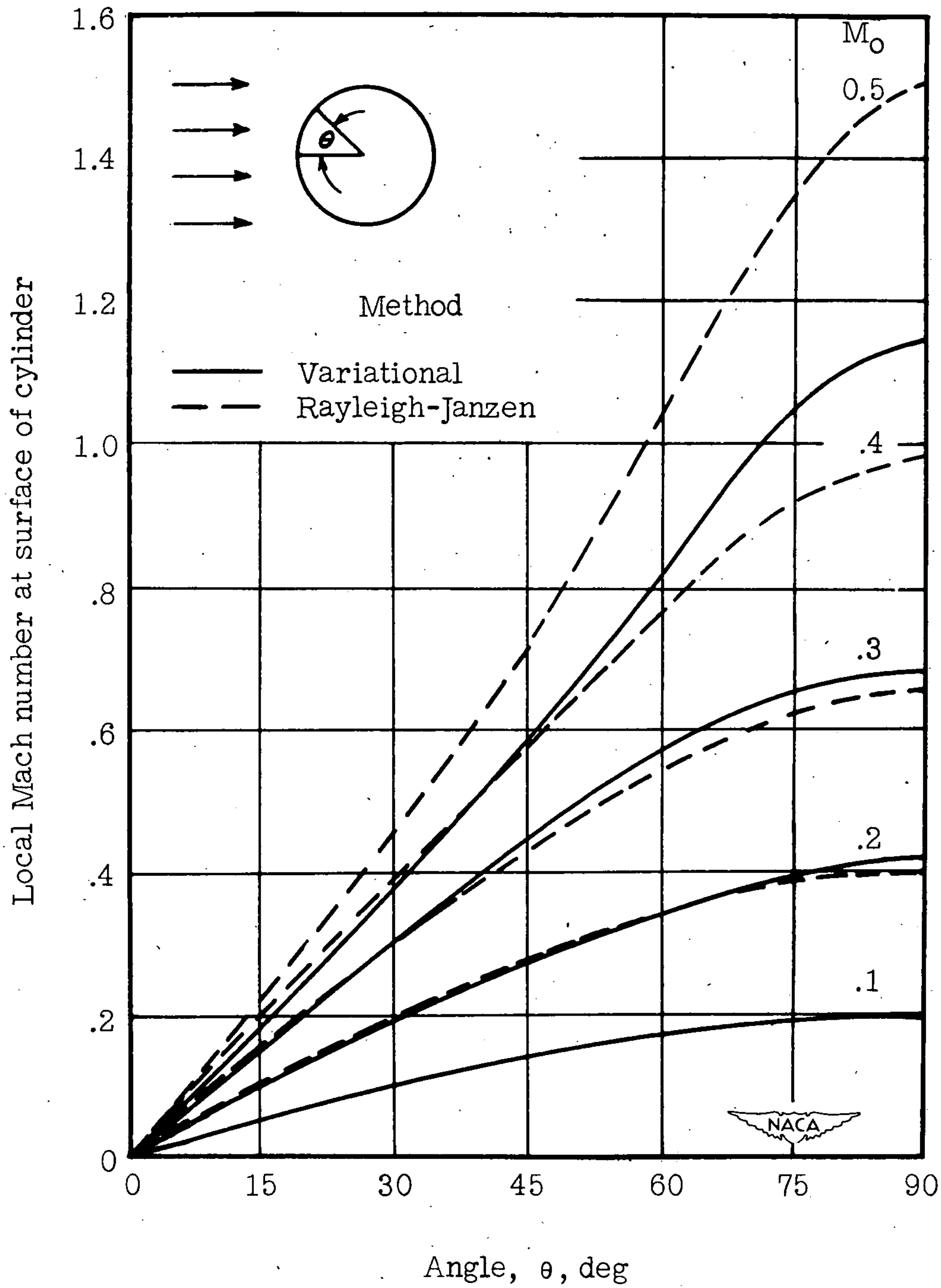

Figure 4.- Local Mach number over surface of cylinder computed by variational and Rayleigh-Janzen methods for various undisturbed-stream Mach numbers.

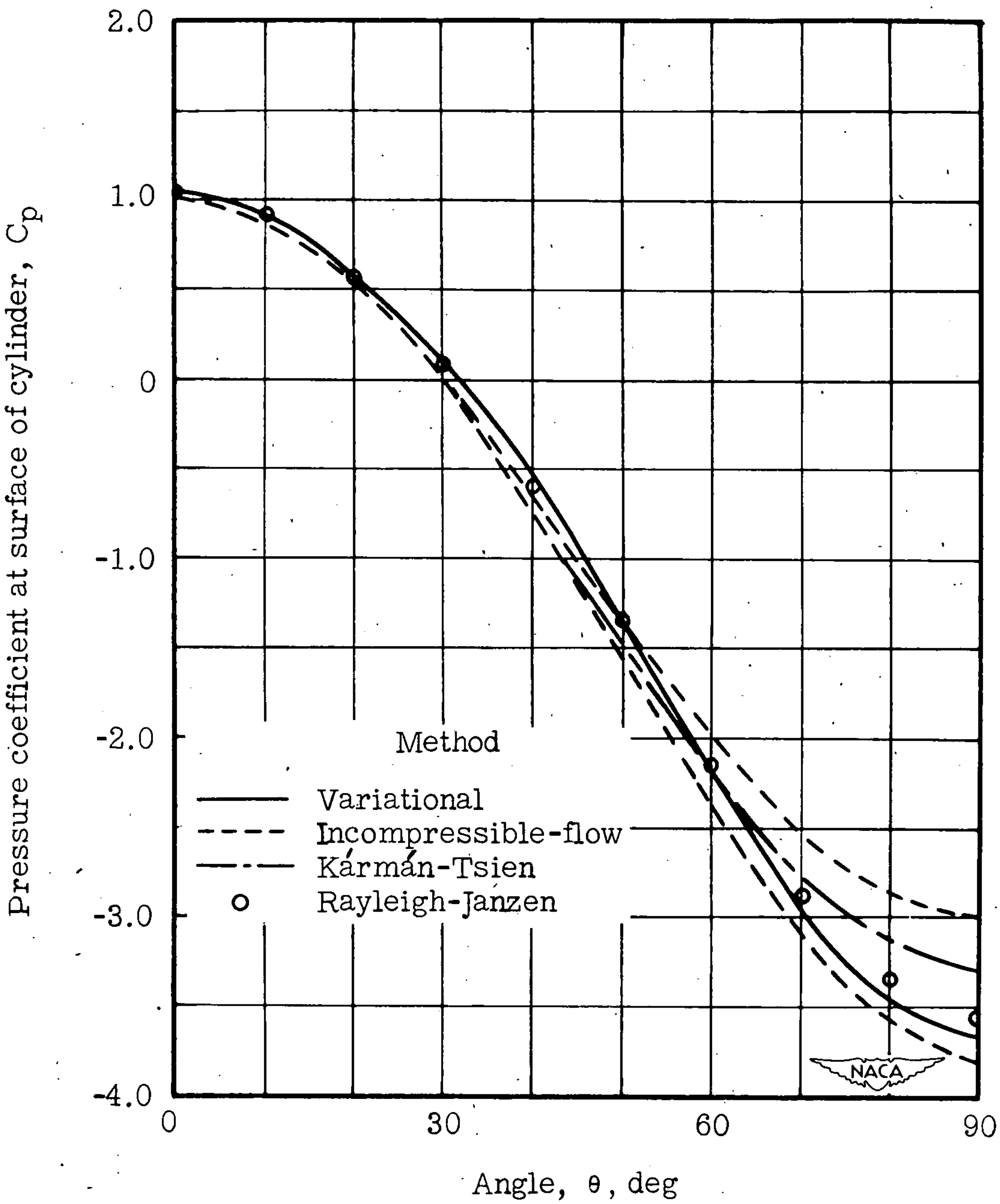

Figure 5.- Pressure coefficient at surface of circular cylinder computed by various methods.

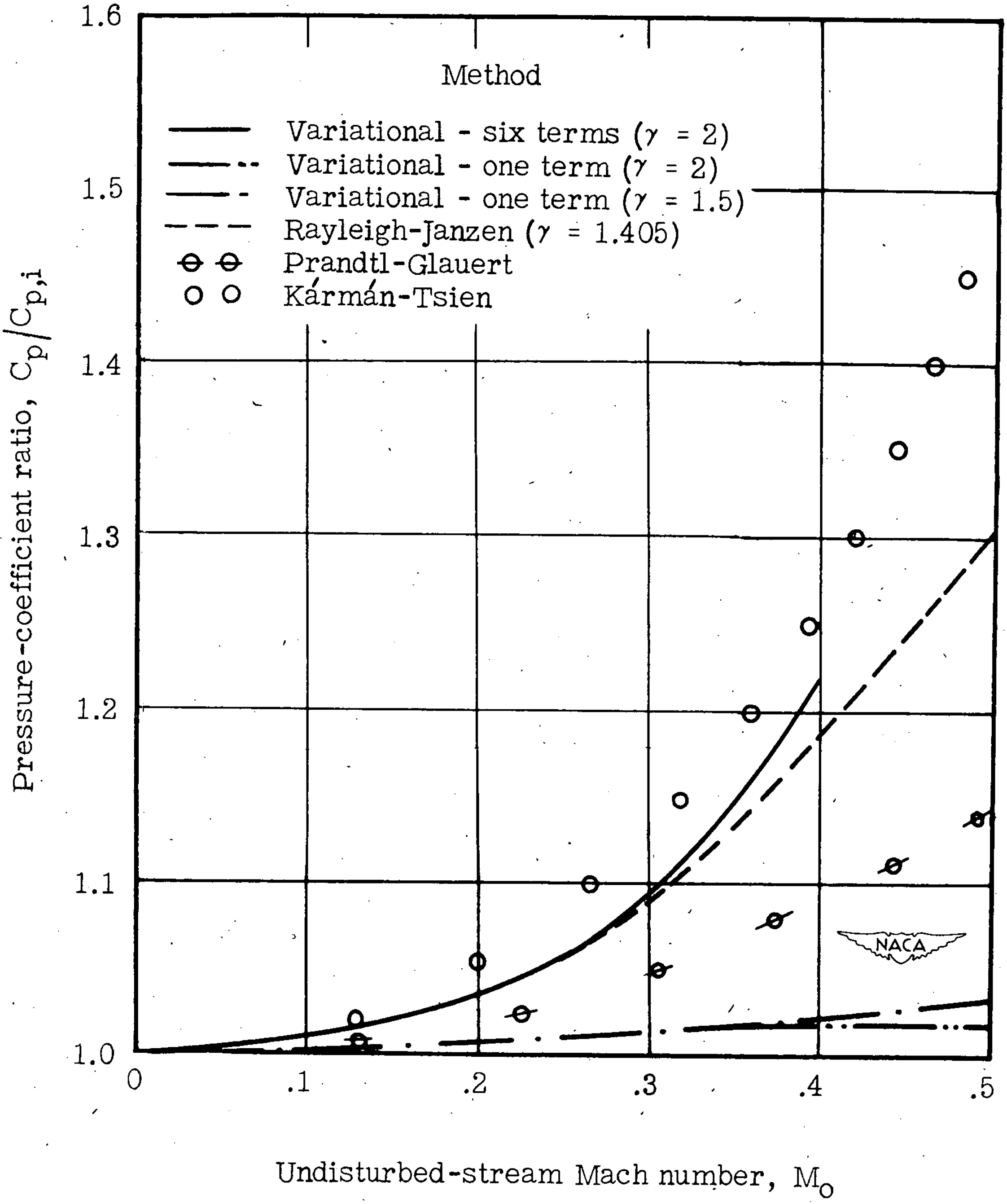

Figure 6.- Pressure-coefficient ratio for circular cylinder at various undisturbed-stream Mach numbers computed by various methods.

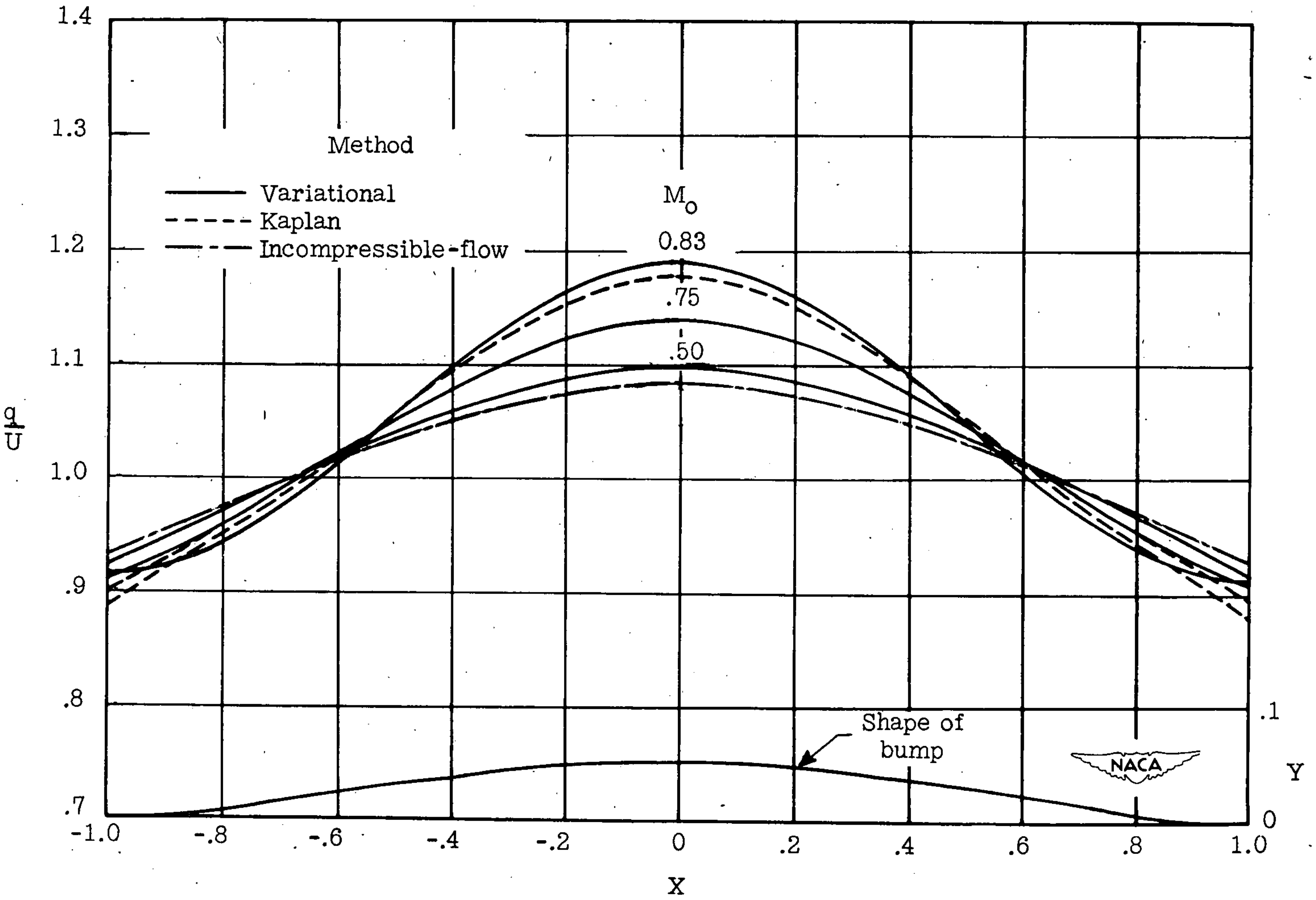

Figure 7.- Shape of bump and velocity distribution over surface of bump computed by variational, Kaplan, and incompressible-flow methods at various undisturbed-stream Mach numbers. Scale at right goes with curve for shape of bump.

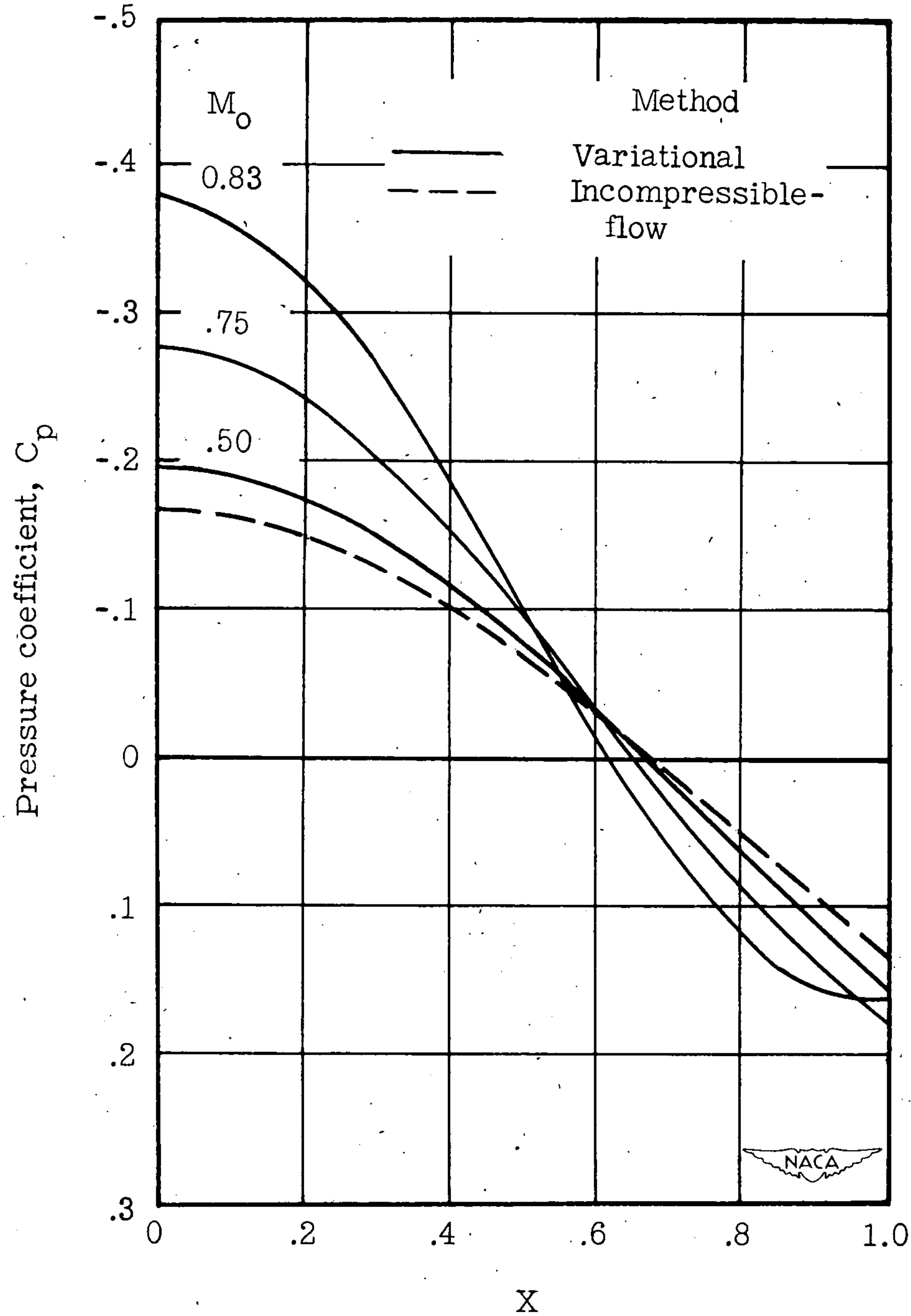

Figure 8.- Pressure coefficients computed by variational and incompressible-flow methods for various undisturbed-stream Mach numbers at various values of X on bump.

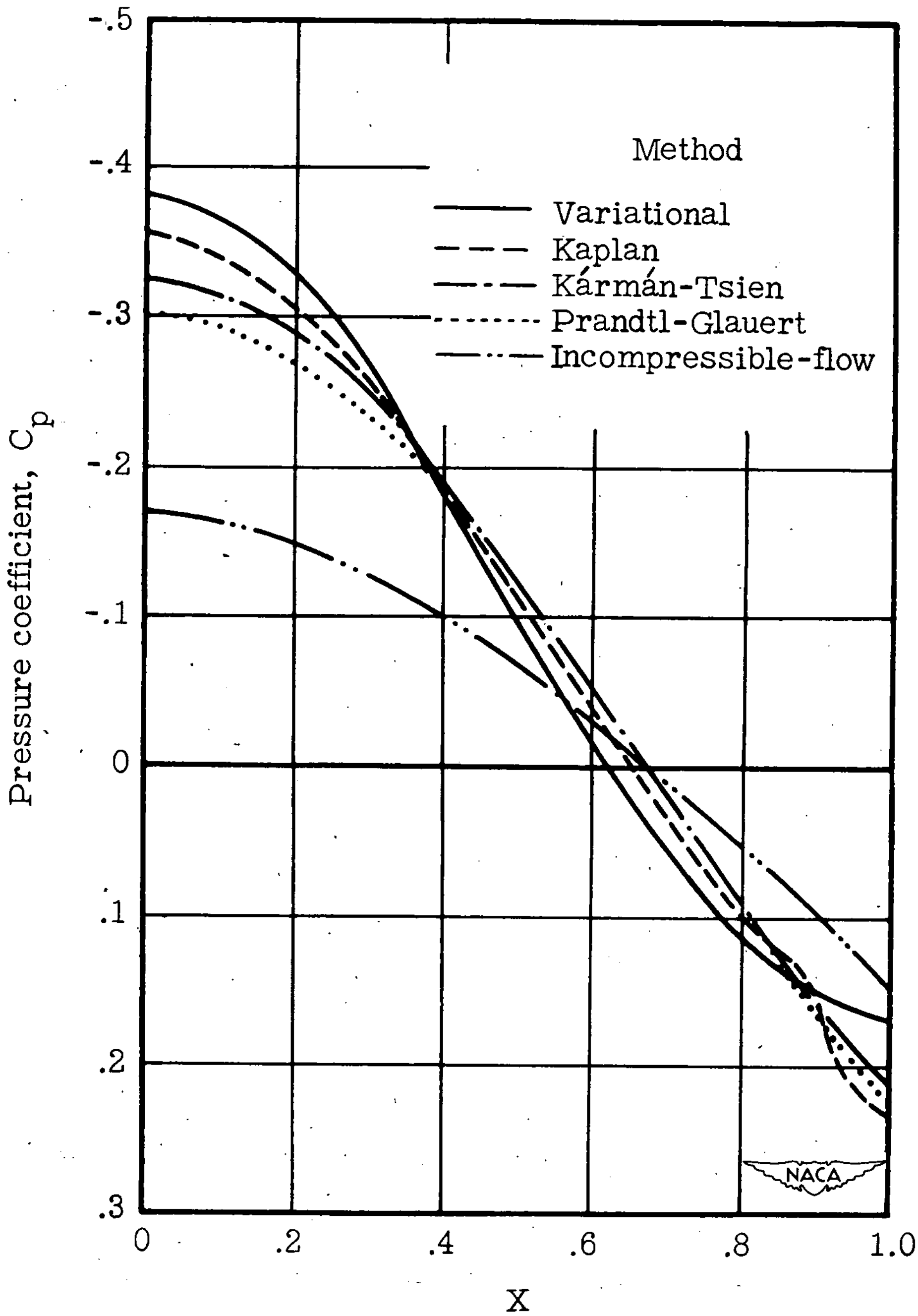

Figure 9.- Pressure coefficients computed by various methods for $M_0 = 0.83$ at various values of X on bump.